Looking at clocks: minimal postulate derivations of space-time and kinematical Lorentz transformations and their physical interpretation

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Abstract

New axiomatic derivations of space-time and kinematical Lorentz transformations are presented. They are based, apart from the usual assumptions of spatial isotropy and homogeneity or single-valuedness, on a simple mathematical postulate, previously tacitly assumed by most authors. Thus neither of Einstein's original postulates are required to derive the Lorentz transformation; the second one is a necessary, though approximate, consequence of relativistic kinematics and the photonic nature of light. The physical interpretation of both space-time and kinematical (velocity) transformations is extensively discussed for specific space-time experiments. Analysis of the Hafele-Keating experiment reveals an important and general misinterpretation of conventional relativistic velocity transformations. Conceptual or mathematical errors in some previous derivations and interpretations of Lorentz transformations are pointed out.

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1 Introduction and overview

Since realising, in 2004 [1], that there are serious conceptual flaws in standard text book presentations of Special Relativity Theory (SRT), the present author has written a series of papers reporting subsequent progress in his understanding of space-time physics. The shortcomings in the standard interpretation are due to insufficient attention to the operational meanings of space and time coordinates in the Lorentz Transformation (LT). In particular, time- and velocity-independent additive constants must be included in order to correctly describe synchronised clocks at different spatial locations. When this is done, it becomes clear that the 'Relativity of Simultaneity' (RS) and 'Length Contraction' (LC) effects of conventional SRT, for which, unlike Time Dilation (TD), there is no experimental evidence, are spurious and unphysical [1, 2]. In three long papers various different aspects of space-time physics were addressed. In the first, [3], measurements of space and time intervals, using a calculus of fundamental pointer-mark coincidences, as well as the related problem of clock synchronisation, without any use of light signals, were considered. In the second, [4], a modern re-appraisal of Einstein's original 1905 special relativity paper [5] is presented. Many errors, both mathematical and conceptual in nature, are uncovered. Einstein's discussion of Classical Electro-Magnetism (CEM), in relation to SRT, in this paper, is also compared to predictions of Relativistic Classical Electro-Dynamics (RCED), a theory recently derived as the classical limit of Quantum Electro-Dynamics (QED) [6]. In the third paper, [7], the important concepts of *physically independent* 'primary' and 'reciprocal' space-time experiments, and of 'base' and 'travelling' frames in such experiments, as well as the transformation formula for relative velocities observed in the same space-time experiment, are introduced, discussed, and applied to Einstein's Train/Embankment [8], and Langevin's 'Travelling Twin' [9] thought experiments. In other papers, an important sign error in Minkowski's original space-time plot [10] that propagated, uncorrected, in text books and the scientific literature throughout the last century is pointed out [11], and more detailed consideration is given to Langevin's thought experiment, in particular the description of the exchange of light signals between the travelling and stay-at-home twins [12]. Various other shorter papers, written by the present author since 2004, and addressing particular topics in space-time physics are cited at the appropriate places below.

The aim of the present paper is a first-principles derivation of all the space-time geometric and kinematical aspects of SRT using the weakest possible postulates as a basis on which to construct the theory. The present author has previously published two different axiomatic derivations of the space-time LT [13, 14]. These derivations are reviewed in the closing section of the present paper and found to be lacking in mathematical or conceptual clarity, insfofar as, in both cases, the essential axiom, called below the 'Inverse Transformation Postulate (ITP), was tacitly assumed by writing down equations that were deemed to be 'obvious'. The style of the present paper is avowedly pedagogical, with a premium on clarity of presentation rather than scientific conciseness. In particular, no attempt is made to avoid repetition of important points, thereby considerably increasing the length of the paper. Following Einstein's adage it is attempted to be 'As simple as possible, but no simpler'. The most important difference between the derivation of the LT presented here, and previous ones, both by the present author and other authors, is that, at every stage of the derivation, all space and time coordinate and velocity symbols have a precise operational definition within a clearly-defined space-time experiment. For this, the previously introduced concepts of primary and reciprocal experiments and base and travelling frames [7] are of crucial importance.

The plan and essential content of the paper are as follows: In the following section, the concepts of primary and reciprocal experiments and the basic postulate (apart from the homogeneity and isotropy of space), from which the LT is derived, is introduced. At this stage, the only modification of Galilean relativity is the replacement of Newtonian absolute time: $T = t' = t$ by the TD relation $t = \gamma(v)t'$ where the 'TD factor', $\gamma(v)$, is some, yet-unknown, even, function of the relative velocity, v, of the two inertial frames S and S′ considered, S being the base and S′ the travelling frame of the space-time experiment. It is also clear that the TD factor is the same in the primary and reciprocal experiments —a condition that is essential for the subsequent derivation of the LT. In Section 3, consideration of the transformation law of the TD factor between three inertial frames S, S′ and S″ determines the TD factor to be $\gamma(v) = 1/\sqrt{1-(v/V)^2}$, where the universal constant, V, is the maximum possible relative velocity of two inertial frames, and the space-time LT is derived. In Section 4, the invariance of the spatial separation of two clocks at rest in the frame S′ is demonstrated and the origin of the spurious RS and LC effects of conventional SRT is explained. Section 5 introduces the Relativistic Velocity Transformation Relation (RRVTR) and contrasts it with the conventional parallel velocity addition relation of SRT, here called the Time Dilatation Velocity Transformation Relation (TDVTR). The symbols v, $\tilde{v}$ and $\hat{v}$ for initial, transformed and 'parametric' velocities are introduced. In Section 6 space-time and kinematical transformations are compared and contrasted. The concepts of base and travelling frames are illustrated by describing observations of clocks at different positions on the surface of the Earth and consideration of the Hafele-Keating experiment [15] in which caesium-beam atomic clocks were flown around the Earth in aircraft in both the East-West and West-East directions before being compared with similar Earth-bound clocks. It is shown that a correct assignment of base and travelling frames is essential in order to correctly describe the observed time intervals recorded in the experiment by the different clocks. The results of the Hafele-Keating experiment also show that the transformation of the velocities of the aircraft into the Earth-centered inertial (base) frame of the experiment is described by the RRVTR, not by the TDVTR. A corollary is the absence of of any LC effect for length intervals on the surface of the Earth as viewed from the Earth-centered inertial frame, thus resolving the Ehrenfest rotating-disc paradox [16]. Relativistic energy and momentum are discussed in Section 7, where it is shown that, if the mass of the photon is very small, relativistic kinematics shows that the speed of light in free space, c, is almost equal to V in all inertial frames (Einstein's second postulate of SRT). It is also shown that relativistic kinematics and the existence of a similar particles with different energies in the same inertial frame excludes a strictly massless nature for the particle. Thus photons with light-like four vectors are incompatible with relativistic kinematics. The final Section 8 contains a summary, discussion and outlook. In the 'discussion' the derivation of the LT in the present paper is compared with two previous ones by the present author [13, 14], as well as those of Ignatowsky [17], Frank and Rohe [18] and Pars [19], all of which are based on simple postulates, without consideration of light signals or any dynamical theory. In the 'outlook' the relevance of the Special Relativity Principle, which is a statement of the inertial-frame-invariance of dynamical laws, for space-time geometry and kinematics, is questioned and it is pointed out that the covariance of Maxwell's Equations, from which Lorentz originally derived the transformation named for him, is only valid to $O(\beta^2)$ in RCED. It is also noted that the universal constant, V, is the limiting velocity only of objects with time-like or light-like four vectors. The virtual photons responsible, in QED, for the inter-charge force have space-like four vectors, which correspond to infinite propagation velocity in the center-of-mass frame of the underlying ee scattering process, consistent with recent measurements of magnetic forces at small source-detector separations [20, 21].

Since all essential concepts and results of this paper are presented, in concise form, in Section 8, the reader with limited time at his or her disposal is recommended, in view of the length of the paper, to first read this section before returning to earlier ones for derivations and more detailed information on particular points of interest.

2 Primary and reciprocal space-time experiments and the time dilation factor $\gamma(v)$

As is well known, linearity of space-time transformation equations is a consequence of the postulate of spatial homogeneity, which requires that the form of the equations remains invariant under the transformations of Cartesian spatial coordinates: $x \rightarrow x + d_x$, $y \rightarrow y + d_y$ and $z \rightarrow z + d_z$ where d_x, d_y and d_z, are arbitary constants [22, 23, 24]. Linearity of the equations may also be derived from the postulates that the world line of an object in uniform motion in free space must be rectilinear in all inertial frames [25], or that the equations must be single-valued functions of their arguments [13]. The latter condition is respected by multilinear transformations[1] The projective transformation required by world-line linearity [25] is a special case of the general multilinear transformation. Linearity follows from the projective or multilinear transformations on requiring, in addition, spatial isotropy: that the form of the equations must not depend on the direction of the spatial coordinate axes. A special case of this is invariance on changing the signs of Cartesian spatial coordinates.

The above considerations lead, in general, to the transformation equations:

$$x' - x'_0 = \gamma(x - x_0) + \rho(t - t_0), \quad (2.1)$$
$$t' - t'_0 = \omega(x - x_0) + \delta(t - t_0), \quad (2.2)$$
$$y' - y'_0 = \phi(y - y_0), \quad z' - z'_0 = \phi(z - z_0) \quad (2.3)$$

with the inverse transformations:

$$x - x_0 = \frac{[\delta(x' - x'_0) - \rho(t' - t'_0)]}{\Delta}, \quad (2.4)$$
$$t - t_0 = \frac{[-\omega(x' - x'_0) + \gamma(t' - t'_0)]}{\Delta}, \quad (2.5)$$
$$y - y_0 = (y' - y'_0)/\phi, \quad z - z_0 = (z' - z'_0)/\phi, \quad (2.6)$$
$$\Delta \equiv \gamma\delta - \omega\rho. \quad (2.7)$$

[1]The general multilinear equation of three variables x, y, z has the form: $x + a_1y + a_2z + b_1xy + +b_2yz + +b_3zx + cxyz = 0$.

The Cartesian spatial coordinates (x,y,z) and (x',y',z') specify the position of the same physical object in two different inertial frames S and S′, respectively, in the case that the coordinate axes of the frames are parallel and that the frames are in uniform relative motion parallel to the x- and x'-axes. The time symbols t (t') are the epochs[2]recorded by similar clocks at rest in S (S′). It will be found convenient, for the following discussion, to introduce two such clocks C and C′ at rest (in general, at arbitary positions) in S and S′ respectively.

Two *physically distinct* space-time experiments may now be considered. In the first *primary* experiment the clock C′ moves with uniform speed v parallel to the x-axis in S. In the second, the clock C moves with uniform speed v' in a direction anti-parallel to the x' axis in S′. This experiment is said to have, relative to the first, a *reciprocal configuration*. In the particular case that $v' = v$ the second experiment is defined to be the *reciprocal* of the primary one. Note that what is given above are only *definitions* of the physically-independent primary and reciprocal experiments. The parameters v and v' are freely chosen initial parameters. In particular, nothing is stated concerning the relation between observations of the same events in the frames S and S′, typically those lying on the world line of some physical object, in either of the two independent experiments.

With these definitions, the coefficients γ, ρ, ω, δ and ϕ in (2.1)-(2.7) in both the primary and reciprocal experiments are either constants or functions of the single initial parameter v. The additive constants x_0, x'_0, y_0, y'_0, z_0 and z'_0 are independent of v but depend on the choice of spatial coordinate systems in S and S′ while t_0 and t'_0 depend on the synchronisation of C and C′. Since C′ is at rest in S′: $x' = x'_0$, and moves along the x-axis with speed v in S, its equation of motion (or world line) in the latter frame may be written as: $x - x_0 = v(t - t_0)$. In this case the transformation equations for the primary experiment are:

$$\mathrm{x}'(\mathrm{C}') = \gamma(v)[\mathrm{x}(\mathrm{C}') - v\mathrm{t}(\mathrm{C})] = 0, \tag{2.8}$$

$$\mathrm{t}'(\mathrm{C}') = \omega(v)\mathrm{x}(\mathrm{C}') + \delta(v)\mathrm{t}(\mathrm{C}), \tag{2.9}$$

$$\mathrm{y}'(\mathrm{C}') = \phi(v)\mathrm{y}(\mathrm{C}'), \quad \mathrm{z}'(\mathrm{C}') = \phi(v)\mathrm{z}(\mathrm{C}') \tag{2.10}$$

where $\mathrm{x}(\mathrm{C}') \equiv x(\mathrm{C}') - x_0(\mathrm{C}')$ etc [3], while those of the reciprocal experiment take the general form:

$$\mathrm{x}(\mathrm{C}) = \gamma(-v)[\mathrm{x}'(\mathrm{C}) + v\mathrm{t}'(\mathrm{C}')] = 0, \tag{2.11}$$

$$\mathrm{t}(\mathrm{C}) = \omega(-v)\mathrm{x}'(\mathrm{C}) + \delta(-v)\mathrm{t}'(\mathrm{C}'), \tag{2.12}$$

$$\mathrm{y}(\mathrm{C}) = \phi(-v)\mathrm{y}'(\mathrm{C}), \quad \mathrm{z}(\mathrm{C}) = \phi(-v)\mathrm{z}'(\mathrm{C}). \tag{2.13}$$

Spatial isotropy, that requires that the equations are invariant under reflection of spatial axes:

$$\mathrm{x} \rightarrow -\mathrm{x}, \quad \mathrm{y} \rightarrow -\mathrm{y}, \quad \mathrm{z} \rightarrow -\mathrm{z}, \quad v \rightarrow -v$$

[2]The word ‘epoch’ is used to denote a particular time registered by a clock. Examples are the number displayed by a digital clock or the number associated with a mark on the face of an analogue clock which is aligned with the hand. A physical time interval recorded by a particular clock is operationally defined as the difference of two epochs registered by the clock.

[3]Use of the roman symbols such as x, t for space or time coordinates implies that no special choice of coordinate systems in the frames S and S’ or of synchronisation of the clocks C and C′ has been made.

gives the following conditions on the v-dependent coefficients:

$$\gamma(v) = \gamma(-v), \tag{2.14}$$
$$\omega(v) = -\omega(-v), \tag{2.15}$$
$$\delta(v) = \delta(-v), \tag{2.16}$$
$$\phi(v) = \phi(-v). \tag{2.17}$$

Some important remarks are now made on the transformation equations (2.8)-(2.10) of the primary experiment and (2.11)-(2.13) of the reciprocal experiment. For the primary experiment, only the spatial cordinates of the clock C′ appear and the position of C is arbitary. The space transformation equation (2.8) is equivalent to the Galilean equations of motion $\mathrm{x}'(\mathrm{C}') = 0$ in S′ and $\mathrm{x}(\mathrm{C}') = v\mathrm{t}(\mathrm{C})$ in S. In fact, $\gamma(v)$ in (2.8) can be replaced by any non-infinite constant without changing the physical significance of the equation. For the reciprocal experiment, only the spatial cordinates of the clock C appear and the position of C′ is arbitary. The space transformation equation (2.11) is equivalent to the Galilean equations of motion $\mathrm{x}(\mathrm{C}) = 0$ in S and $\mathrm{x}'(\mathrm{C}) = -v\mathrm{t}'(\mathrm{C}')$ in S′.

Some general kinematical configurations of the clocks in S and S′ in the primary and reciprocal experiments are shown in Fig. 1a. In Fig. 1b, without entailing any loss of generality in the description of the problem, are shown configurations in S and S′ at corresponding epochs in the primary and reciprocal experiments at which the clock C is chosen to have the same x, x' and y, y' coordinates as C′ in the primary experiment and C′ is chosen to have the same x, x' and y, y' coordinates as C in the reciprocal experiment. With this choice of clock positions it follows that, at the epochs shown in Fig. 1b:

$$\mathrm{x}(\mathrm{C}') = \mathrm{x}(\mathrm{C}), \quad \mathrm{x}'(\mathrm{C}') = \mathrm{x}'(\mathrm{C}) \tag{2.18}$$

while, at all times

$$\mathrm{y}(\mathrm{C}') = \mathrm{y}(\mathrm{C}), \quad \mathrm{y}'(\mathrm{C}') = \mathrm{y}'(\mathrm{C}). \tag{2.19}$$

Combining (2.10),(2.13),(2.17) and (2.19) gives, at all times,

$$\mathrm{y}'(\mathrm{C}') = \phi(v)\mathrm{y}(\mathrm{C}) = \phi(v)\phi(-v)\mathrm{y}'(\mathrm{C}') = \phi(v)^2\mathrm{y}'(\mathrm{C}') \tag{2.20}$$

so $\phi(v)^2 = 1$, and since the y- and y'-axes are chosen parallel, $\phi(v) = 1$.

Substituting (2.18) into the space transformation equation of the reciprocal experiment, and using (2.14), leads to the suggestive relation:

$$\mathrm{x}(\mathrm{C}') = \gamma(v)[\mathrm{x}'(\mathrm{C}') + v\mathrm{t}'(\mathrm{C}')]. \tag{2.21}$$

However, (2.11) and (2.18) give $\mathrm{x}'(\mathrm{C}) = -v\mathrm{t}'(\mathrm{C}')$ and $\mathrm{x}(\mathrm{C}) = \mathrm{x}(\mathrm{C}') = 0$. Setting then $\mathrm{x}'(\mathrm{C}') = 0$ in (2.21) implies only that $\mathrm{t}'(\mathrm{C}') = 0$ at the epoch of C′ shown in Fig. 1b. Then Eq. (2.21) *per se* provides no useful constraint on the coefficients of the transformation equations.

At this point, the only postulate, in addition to the homogeneity and isotropy of space, to be used in the present derivation of the transformation equations is introduced. It may be called the 'Inverse Transformation Postulate' (ITP). This postulate asserts that that

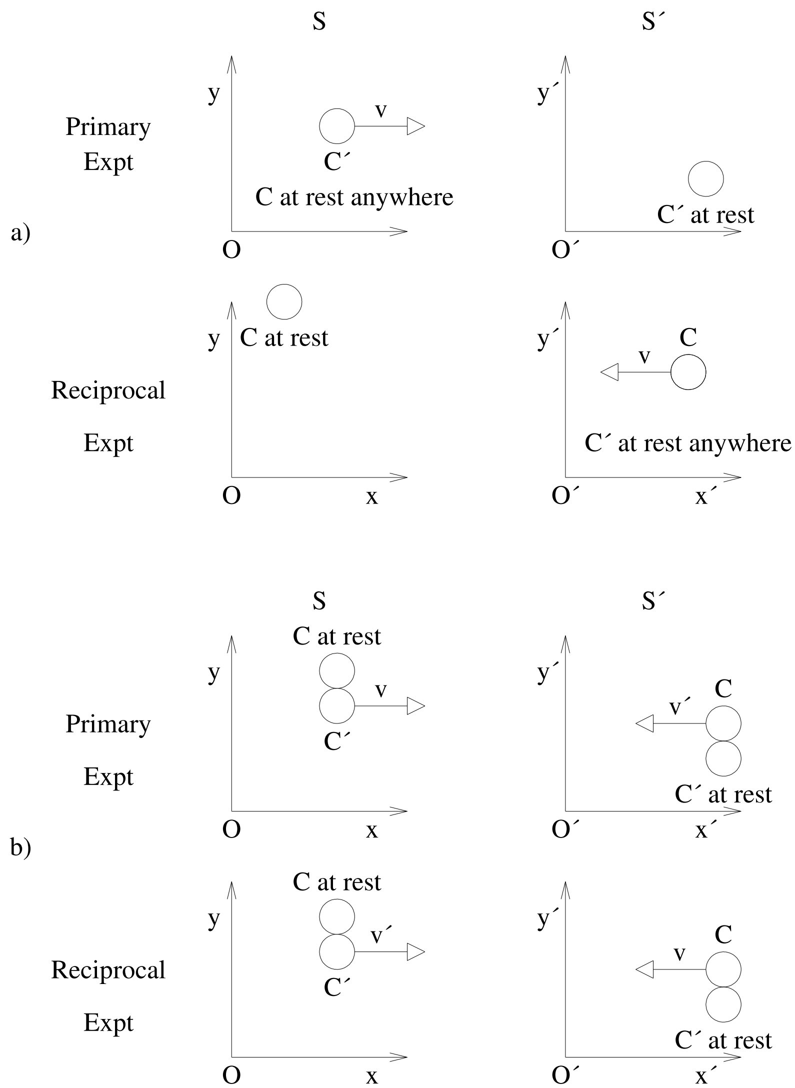

Figure 1: *Spatial configurations of the clocks C and C' in primary and reciprocal experiments. a) General case. b) At the epoch at which C and C' are aligned in the x-direction, and the* y,y' *and* z,z' *coordinates of C and C' are equal. The velocity* v' *is given as* $v\gamma(\beta_v)$ *by the RRVTR Eq. (5.4). See text for discussion.*

(2.21), valid at the particular epoch $\mathrm{t}'(\mathrm{C}') = 0$ in the reciprocal experiment, *holds also in the primary experiment for all values of* $\mathrm{t}'(\mathrm{C}')$, and that the corresponding relation:

$$\mathrm{x}'(\mathrm{C}) = \gamma(v)[\mathrm{x}(\mathrm{C}) - v\mathrm{t}(\mathrm{C})] \tag{2.22}$$

holds, *for all values of* $\mathrm{t}(\mathrm{C})$, *in the reciprocal experiment.* This postulate is equivalent to setting $\delta = \gamma$, $\Delta = 1$ in the inverse transformation equation equations (2.3) and (2.4), so that the space and time coordinates are treated in a symmetrical manner. Now setting $\mathrm{x}'(\mathrm{C}') = 0$ in (2.21), on the assumption that it holds true for the primary experiment, gives, together with (2.8):

$$\mathrm{x}(\mathrm{C}') = v\mathrm{t}(\mathrm{C}) = \gamma(v)v\mathrm{t}'(\mathrm{C}') \tag{2.23}$$

which yields a *spatial coordinate-independent* time dilation (TD) relation:

$$\mathrm{t}(\mathrm{C}) = \gamma(v)\mathrm{t}'(\mathrm{C}'). \tag{2.24}$$

Combining in a similar manner (2.22) with (2.11) and (2.14) gives the TD relation for the reciprocal experiment:

$$\mathrm{t}'(\mathrm{C}') = \gamma(v)\mathrm{t}(\mathrm{C}). \tag{2.25}$$

To avoid confusion between epoch symbols in the physically distinct primary and reciprocal experiments[4] these TD relations may be rewritten as:

Primary Experiment

$$\mathrm{t}(\mathrm{C}') \equiv \mathrm{t}(\mathrm{C}) = \gamma(v)\mathrm{t}'(\mathrm{C}') \equiv \gamma(v)\tau(\mathrm{C}'), \tag{2.26}$$

Reciprocal Experiment

$$\mathrm{t}'(\mathrm{C}) \equiv \mathrm{t}'(\mathrm{C}') = \gamma(v)\mathrm{t}(\mathrm{C}) \equiv \gamma(v)\tau(\mathrm{C}). \tag{2.27}$$

These equations introduce the concepts of 'proper time', $\tau(\mathrm{C}')$,of a clock C' as observed in its own rest frame, S', and $\mathrm{t}(\mathrm{C}')$,which is the corresponding 'improper time' of the same clock when observed from an inertial frame S, relative to which it is motion. In any actual experiment, as explicitly shown in (2.26) and (2.27), the 'improper time' of the moving clock is operationally defined as the epoch recorded by a similar clock which is at rest in the frame from which the moving clock is observed.

Without any further calculation, the essential structure of the space-time transformation equations between two inertial frames is already established:

Primary Experiment (Lorentz)

$$\mathrm{x}'(\mathrm{C}') = 0, \quad \mathrm{x}(\mathrm{C}') = v\mathrm{t}(\mathrm{C}'), \tag{2.28}$$

$$\mathrm{t}(\mathrm{C}) = \mathrm{t}(\mathrm{C}') = \gamma(v)\tau(\mathrm{C}'), \tag{2.29}$$

Reciprocal Experiment (Lorentz)

$$\mathrm{x}(\mathrm{C}) = 0, \quad \mathrm{x}'(\mathrm{C}) = -v\mathrm{t}'(\mathrm{C}), \tag{2.30}$$

$$\mathrm{t}'(\mathrm{C}') = \mathrm{t}'(\mathrm{C}) = \gamma(v)\tau(\mathrm{C}) \tag{2.31}$$

[4]Unless this is done (2.24) and (2.25) require, algebraically, that $\mathrm{t} = \gamma\mathrm{t}' = \gamma^2 t$ or $\gamma^2 = 1$, which necessarily gives Galilean transformation equations.

where the improper times $\mathrm{t(C')}$ and $\mathrm{t'(C)}$ are measured by clocks at rest in the other inertial frame: $\mathrm{t(C')} = \mathrm{t(C)}$ and $\mathrm{t'(C)} = \mathrm{t'(C')}$. Newtonian universal time: $\mathrm{t} = \mathrm{t'} = \mathrm{T}$ corresponds to $\gamma(v) = 1$ and the Galilean transformations:

Primary Experiment (Galilean)

$$\mathrm{x'(C')} = 0, \quad \mathrm{x(C')} = v\mathrm{T}, \tag{2.32}$$

$$\mathrm{t}(C) = \mathrm{t(C')} = \mathrm{T}, \tag{2.33}$$

Reciprocal Experiment (Galilean)

$$\mathrm{x(C)} = 0, \quad \mathrm{x'(C)} = -v\mathrm{T}, \tag{2.34}$$

$$\mathrm{t'(C')} = \mathrm{t'(C)} = \mathrm{T}. \tag{2.35}$$

The equations in (2.32) and (2.34) can be written in an equivalent manner as:

$$\mathrm{x'(C')} = \mathrm{x(C')} - v\mathrm{T} = 0, \tag{2.36}$$

$$\mathrm{x(C)} = \mathrm{x'(C)} + v\mathrm{T} = 0. \tag{2.37}$$

Transposing the first members of these equations gives

$$\mathrm{x(C')} = \mathrm{x'(C')} + v\mathrm{T}, \tag{2.38}$$

$$\mathrm{x'(C)} = \mathrm{x(C)} - v\mathrm{T} \tag{2.39}$$

which are the Galilean ($\gamma \to 1$, $\mathrm{t'(C')}$, $\mathrm{t(C)} \to \mathrm{T}$) limits of the postulated Lorentz transformation equations (2.21) and (2.22) respectively. The latter are thus consistent with Galilean equations of motion for $\gamma = 1$.

Comparing the special relativistic transformation formulae (2.28)-(2.31) with the Galilean ones (2.32)-(2.35) it can be seen that the space transformation equations describing the equations of motion (or worldlines) of C and C′ in S and S′ in the primary and reciprocal experiments are identical in the two cases. Only the time transformation formula $\mathrm{t'} = \mathrm{t} = \mathrm{T}$ of Newtonian physics is modified by the TD relations (2.29) and (2.31) which show that the time transformation is given by a universal scale factor $\gamma(v)$ for a given pair of inertial frames, without any dependence on the spatial positions of the two clocks. A corollary of this is the absence of any spatially-dependent 'relativity of simultaneity' (RS) effect as suggested by Einstein [5]. The TD relations for two clocks $\mathrm{C'_1}$, $\mathrm{C'_2}$ in the primary experiment are:

$$\begin{aligned} \mathrm{t(C'_1)} &= \gamma(v)\tau(\mathrm{C'_1}), & (2.40)\\ \mathrm{t(C'_2)} &= \gamma(v)\tau(\mathrm{C'_2}). & (2.41) \end{aligned}$$

It is obvious that, if the clocks are synchronised so that $\tau(\mathrm{C'_1}) = \tau(\mathrm{C'_2})$, then also $\mathrm{t(C'_1)} = \mathrm{t(C'_2)}$, so that the clocks are seen to be synchronous also in the frame S. There is no RS effect depending on the spatial separation of the clocks. How the spurious prediction of such an effect arises in conventional special relativity theory is explanined in Refs. [2, 26, 27, 28], and is recalled in Section 4 below.

3 Determination of the time dilation factor and its transformation law. The limiting velocity V

The space-time transformation equations have now been determined up to one unknown function $\gamma(v)$ which is an even function of the initial velocity parameter v. To proceed further, the first step is to derive a time transformation equation by eliminating x(C′) between Eqs. (2.8) and (2.21), or x′(C) between Eqs. (2.11) and (2.22). This gives, for the primary experiment:

$$\mathrm{t(C)} = \gamma(v)\left[\mathrm{t'(C')} + \frac{\eta(v)}{v}\mathrm{x'(C')}\right], \qquad (3.1)$$

where

$$\eta(v) \equiv \frac{\gamma(v)^2 - 1}{\gamma(v)^2}. \qquad (3.2)$$

Inspection of the TD relation shows that since, when $v = 0$, the frames S and S′ become identical, so that $\mathrm{t} = \mathrm{t'}$, then $\gamma(v = 0) = 1$. In general, $\gamma(v)$, at least in the region of $v = 0$, must be either an increasing or a decreasing function of v. In the former case $\gamma(v) > 1$ in the latter $\gamma(v) < 1$, these inequalities holding for all values of v if γ is a monotonic function of v. Assuming this to be the case, the following inequalities may be derived from (3.2):

$$1 \geq \eta(v) > 0 \quad [\gamma(v) > 1], \qquad (3.3)$$
$$0 > \eta(v) \geq -1 \quad [\gamma(v) < 1] \qquad (3.4)$$

so that $\eta(v)$ is an increasing (decreasing) function of v when $\gamma(v)$ is an increasing (decreasing) function of v. Consider now the velocity, V, defined as the solution of the equation $\eta(V) = 1$ in (3.3) or of $\eta(V) = -1$ in (3.4). If such a solution exists, then V is the maximum allowed relative velocity of two inertial frames, and as such must be a universal constant. As will be seen, the existence of a monotonically increasing function $\eta(v)$ with a limiting velocity given by $\eta(V) = 1$ is required by self-consistency of transformation equations of the TD factor γ. It will be demonstrated, in Section 7 below, that the speed of light in free space, c, is equal, to a very good approximation, to the limiting velocity V in all inertial frames, as a consequence of relativistic kinematics and the photonic nature of light. In this way Einstein's second postulate of special relativity is *derived*, from first principles, not assumed *a priori*.

The transformation of intervals $\Delta\mathrm{x'}$, $\Delta\mathrm{t'}$ between events on the world lines of two points on the x' axis is now considered. The corresponding interval $\Delta\mathrm{t}$ in the frame S is given by (3.1) as:

$$\Delta\mathrm{t(C)} = \gamma(v)\left[\Delta\mathrm{t'(C')} + \frac{\eta(v)}{v}\Delta\mathrm{x'}\right]. \qquad (3.5)$$

Consider now a third clock, C″, at rest in the frame S″, that moves with speed u' parallel to the x'-axis in S′. Assuming that $\Delta\mathrm{x'}$ is an interval, $\Delta\mathrm{x'(C'')}$, on the world line of the clock then:

$$u' = \frac{\Delta\mathrm{x'(C'')}}{\Delta\mathrm{t'(C')}}. \qquad (3.6)$$

The TD relation between the frames S′ and S″ is:

$$\Delta t'(C') = \gamma(u')\Delta\tau(C''). \tag{3.7}$$

With the aid of (3.6) and (3.7), (3.5) may be written as:

$$\frac{\Delta t(C)}{\Delta\tau(C'')} \equiv \gamma(u) = \gamma(v)\gamma(u')\left[1 + \frac{\eta(v)}{v}u'\right], \tag{3.8}$$

where $\gamma(u)$ is the TD factor between the frames S and S″:

$$\Delta t(C) = \gamma(u)\Delta\tau(C''). \tag{3.9}$$

The formula, analogous to (3.5), giving the time interval $\Delta t''$ corresponding to $\Delta x'$ and $\Delta t'$ is

$$\Delta t''(C'') = \gamma(u')\left[\Delta t'(C') - \frac{\eta(u')}{u'}\Delta x'\right] \tag{3.10}$$

If $\Delta x'$ is now an interval, $\Delta x'(C)$, on the world line in S′ of the clock C, then Eq. (2.30) gives:

$$\frac{\Delta x'(C)}{\Delta t'(C')} = -v \tag{3.11}$$

In view of the TD relation (see Eq (2.31)),

$$\Delta t'(C') = \gamma(v)\Delta\tau(C) \tag{3.12}$$

(3.10) and (3.11) give a relation similar to (3.8):

$$\frac{\Delta t''(C'')}{\Delta\tau(C)} \equiv \gamma(\bar{u}) = \gamma(u')\gamma(v)\left[1 + \frac{\eta(u')}{u'}v\right]. \tag{3.13}$$

The TD relation in the first member of (3.13):

$$\Delta t''(C'') = \gamma(\bar{u})\Delta\tau(C) \tag{3.14}$$

is that of the experiment reciprocal to the one with the TD relation of (3.9). However, (2.24) and (2.25) above show that the TD factors of an experiment and its reciprocal are equal:

$$\gamma(\bar{u}) = \gamma(u), \qquad \bar{u} = u. \tag{3.15}$$

Taking into consideration (3.15) it follows from (3.8) and (3.13) that:

$$\frac{\eta(v)}{v}u' = \frac{\eta(u')}{u'}v \tag{3.16}$$

or

$$\frac{\eta(v)}{v^2} = \frac{\eta(u')}{(u')^2} = \pm\frac{1}{V^2} \tag{3.17}$$

where V is the solution of the equation $\eta(V) = 1$, or $\eta(V) = -1$, discussed above. That the ratios in the first member of (3.17) must be equal to some constant follows also from the independence of the variables v and u'. It is demonstrated in Ref. [13] that a negative sign for the last member of (3.17), implying that $\gamma(v) < 1$, is excluded by the condition of single-valuedness of the transformation equations. Other arguments to reject the case

$\gamma(v) < 1$ may be found in Ref. [29]. The TD factor is now determined by Eqs (3.2) and (3.17) to be the monotonically increasing function of v:

$$\gamma(v) = \gamma(\beta_v) \equiv \frac{1}{\sqrt{1-\beta_v^2}}, \tag{3.18}$$

where

$$\beta_v \equiv v/V = [\eta(v)]^{\frac{1}{2}}\,. \tag{3.19}$$

Using (3.18) and (3.19) the transformation equation (3.8) of the TD factor $\gamma(\beta_{u'})$ between the frames S′ and S may be written as

$$\begin{aligned}\gamma(\beta_u) &= \gamma(\beta_v)[\gamma(\beta_{u'}) + \beta_v\beta_{u'}\gamma(\beta_{u'})], \\ &= \gamma(\beta_v)\gamma(\beta_{u'})[1+\beta_v\beta_{u'}]\end{aligned} \tag{3.20}$$

to be compared with the time transformation equation between the same two frames given by (3.1), (3.2) and (3.18):

$$\mathrm{x}_0 \equiv V\mathrm{t} = \gamma(\beta_v)[V\mathrm{t}' + \beta_v\mathrm{x}'] \equiv \gamma(\beta_v)[\mathrm{x}_0' + \beta_v\mathrm{x}']. \tag{3.21}$$

Comparison of (3.20) and (3.21) shows that $(\gamma(\beta_{u'}), \beta_{u'}\gamma(\beta_{u'}))$ transforms in the same way as $(\mathrm{x}_0', \mathrm{x}')$ between the frames S′ and S. Combining (3.18) with (3.20) gives, for $\beta_u > 0$:

$$\begin{aligned}\beta_u &= \left[1-\frac{1}{\gamma(\beta_u)^2}\right]^{\frac{1}{2}} = \left[1-\frac{1}{\gamma(\beta_v)^2\gamma(\beta_{u'})^2(1+\beta_v\beta_{u'})^2}\right]^{\frac{1}{2}} \\ &= \left[1-\frac{(1-\beta_v^2)(1-\beta_{u'}^2)}{(1+\beta_v\beta_{u'})^2}\right]^{\frac{1}{2}} = \frac{\beta_v+\beta_{u'}}{1+\beta_v\beta_{u'}}.\end{aligned} \tag{3.22}$$

This is the well-known parallel velocity addition relation of special relativity derived by Einstein in 1905 [5]. Note, however, the physical meaning of (3.22). It is the transformation law of the velocity specifiying, via Eq (3.18), the TD factor, for an experiment in which the moving clock C″ is observed from the frame S, given the initial velocities v and u'. Alternatively, this TD factor is given directly by the transformation equation (3.20). The velocity β_u is then a parameter that may be used to specify the TD factor. However β_u in (3.22) *does not describe the observed velocity of* C″ *in the frame S*, which is how it has hitherto been interpreted in conventional special relativity theory. The correct formula for the observed velocity of C″ in S in the primary experiment is derived in Section 5 below. Use of (3.20) and (3.22) leads to a further, but algebraically equivalent, transformation relation:

$$\gamma(\beta_v)[\beta_{u'}\gamma(\beta_{u'}) + \beta_v\gamma(\beta_{u'})] \equiv \gamma(\beta_v)\gamma(\beta_{u'})[\beta_{u'}+\beta_v] = \frac{\gamma(\beta_u)[\beta_{u'}+\beta_v]}{1+\beta_v\beta_{u'}} = \beta_u\gamma(\beta_u) \tag{3.23}$$

which has the same structure as the space transformation equation:

$$\mathrm{x} = \gamma(\beta_v)[\mathrm{x}' + \beta_v\mathrm{x}_0'] \tag{3.24}$$

showing again that $(\gamma(\beta_{u'}), \beta_{u'}\gamma(\beta_{u'}))$ transforms in the same way as $(\mathrm{x}_0', \mathrm{x}')$ between the frames S′ and S. This circumstance enables $\gamma(\beta_{u'})$ and $\beta_{u'}\gamma(\beta_{u'})$ to be identified as the temporal and spatial components, respectively, of the dimensionless four-vector velocity $U' \equiv (U'_0; U'_x, 0, 0)$ of the clock C″ in the frame S′:

$$U'_0 \equiv \gamma(\beta_{u'}), \quad (3.25)$$
$$U'_x \equiv \beta_{u'}\gamma(\beta_{u'}) \quad (3.26)$$

that, in virtue of the identity: $\gamma(\beta_{u'})^2 - \beta_{u'}^2\gamma(\beta_{u'})^2 \equiv 1$, yields the invariant four-vector scalar product:

$$U' \cdot U' \equiv (U'_0)^2 - (U'_x)^2 = 1. \quad (3.27)$$

As will be seen in Section 7 below, Eq. (3.27) is the basis of the relation connecting the relativistic energy and momentum of a physical object with its Newtonian mass.

4 Spatially-separated synchronised clocks and the invariance of spatial intervals

The discussion of the previous section concerning equations of motion and velocities of objects in the frames S and S′, as well as the transformation law of time intervals between these frames, made no assumption concerning the choice of spatial coordinate systems and clock synchronisation in the two frames which are specified by the velocity-independent constants x_0, x'_0, y_0, y'_0, z_0, z'_0, t_0 and t'_0. These constants may be freely chosen without modifying any physical predictions. With the particular choice: $x_0 = x'_0 = D'$, $t_0 = t'_0 = 0$, The transformation equations in primary experiment, (2.28) and (2.29) become[5]

$$x'(\mathrm{C}') = D', \quad x(\mathrm{C}') = D' + vt(\mathrm{C}), \quad (4.1)$$

$$t(C) = t(\mathrm{C}') = \gamma(\beta_v)\tau(\mathrm{C}') \quad (4.2)$$

and those, (2.30), (2.31) for the reciprocal experiment are:

$$x(\mathrm{C}) = D', \quad x'(\mathrm{C}) = D' - vt'(\mathrm{C}'), \quad (4.3)$$

$$t'(\mathrm{C}') = t'(\mathrm{C}) = \gamma(\beta_v)\tau(\mathrm{C}). \quad (4.4)$$

It can be seen from these equations that in the primary experiment when $x'(\mathrm{C}') = x(\mathrm{C}') = D'$ then $t(C) = \tau(\mathrm{C}') = 0$, and in the reciprocal experiment when $x(\mathrm{C}) = x'(\mathrm{C}) = D'$ then $t'(C') = \tau(\mathrm{C}) = 0$, so, both cases, the clocks C and C′ are synchronised.

Using the same coordinate systems in S and S′ as in (4.1)-(4.4) and introducing two synchronised clocks C'_1, C'_2, at rest in S′ on the x'-axis at $x' = D'_1, D'_2$ respectively with epochs: $t'(\mathrm{C}'_1) = t'(\mathrm{C}'_2) = \tau'$, gives the transformation equations for the primary experiment:

$$x'(\mathrm{C}'_1) = D'_1, \quad x(\mathrm{C}'_1) = D'_1 + vt(\mathrm{C}), \quad (4.5)$$

[5]The meaning of the symbol τ (defined with, or without, an arbitary constant subtracted from the current clock epoch) is evident on inspection of the other time coordinate symbols of the equation in which it occurs. Note that, from this point on, mathmode coordinate symbols are employed to indicate that a particular choice of spatial coordinates and clock synchronisation constants has been made. See the following discussion of Fig. 2.

$$t(C) = t(C'_1) = \gamma(\beta_v)\tau', \tag{4.6}$$

$$x'(C'_2) = D'_2, \quad x(C'_2) = D'_2 + vt(C), \tag{4.7}$$

$$t(C) = t(C'_2) = \gamma(\beta_v)\tau'. \tag{4.8}$$

Eqs. (4.6) and (4.8) give $t(C'_1) = t(C'_2) = t(C) = \gamma(\beta_v)\tau'$ so that C'_1 and C'_2 are observed to be synchronous (record the same epoch) also in the frame S, for for arbitary values of τ' . There is therefore no RS effect. Eqs. (4.5) and (4.7) give, for all all values of the epoch $t(C)$,

$$x'(C'_2) - x'(C'_1) \equiv L' = D'_2 - D'_1 = x(C'_2) - x(C'_1) \equiv L \tag{4.9}$$

—there is no 'length contraction' (LC) effect.

Before examining how the spurious and correlated RS and LC effects arise in conventional special relativity theory, it is instructive to consider in detail how the particular choice of spatial coordinate systems and clock synchronisation constants embodied in Eqs (4.1)-(4.8) is related to the general, and completely arbitary, choice of these constants that can be made without changing the predictions for any observable physical effect. Since the space-time LT describes only the transformation laws of physical space and time *intervals* and since, as discussed in Ref. [30] a physical spatial interval, unlike a clock epoch, a time interval or a spatial coordinate, is an attribute of *two* physical objects, it will be found convenient to introduce two 'reference objects' R and R′ at initially arbitary positions on the $x-$ and $x'-$ axes, respectively, in order to define spatial intervals in the frames S and S′. Consider first two arbitary coordinates x, x' both specifed in the rest frame of the clock C′, as shown in Fig. 2a. Since the positions of the coordinate origins are arbitary, the numbers representing the coordinates of, say, R, are also arbitary. However the physical spatial intervals, D, D' between R and C′, R′ and C′, respectively, do not depend on the choice of coordinate system i.e. the positions of the coordinate origins O and O′. The x and x' coordinates of a particular object at any position can then be, in general, any real numbers whatsoever.

In Fig 2b, R is moved into spatial coincidence with R′. In this case $D = D'$ and the x or x' coordinates of R and R′ must be equal, but may separately take any value. In Fig. 2c the coordinate origin O is chosen to be aligned with R and O′ to be aligned with R′. Now the x coordinate of R is zero, as is the x' coordinate of R′, while the x and x' coordinates of C′ are both equal to D'. The similar rulers Ru and Ru′ measuring, respectively the spatial separations of R and C′ and R′ and C′, respectively, are also introduced in Fig. 2c. The object R is aligned with first mark of Ru and C′ with the sixteenth mark of this ruler Similarly R′, C′ are aligned with the first and sixteenth marks of Ru′. In Fig.2d, at the epochs $t(C) = t'(C') = 0$, an equal impulsive acceleration occupying a negligibly short time interval, and during which the accelerated objects undergo negligible spatial displacements, is applied to R′, C′ and Ru′. After this acceleration, the proper frame of the accelerated objects is the inertial frame S′, considered above that is moving with speed v along the x axis, while R and Ru are at rest in the frame S. It is clear, from inspection of Fig. 2c and Fig. 2d, that immediately following the impulsive acceleration, the spatial separation, D', of R′ and C′ in S' is equal to that of R and C′, or R′ and C′, in S, in accordance with Eqs (4.1) and (4.2) above. It then follows from these equations that the equality $D = D'$ where now $D \equiv x(C') - x(R')$ holds for all later epochs $t(C)$. The configuration in Fig. 2d also corresponds to that of the clocks C'_1 and C'_2 described

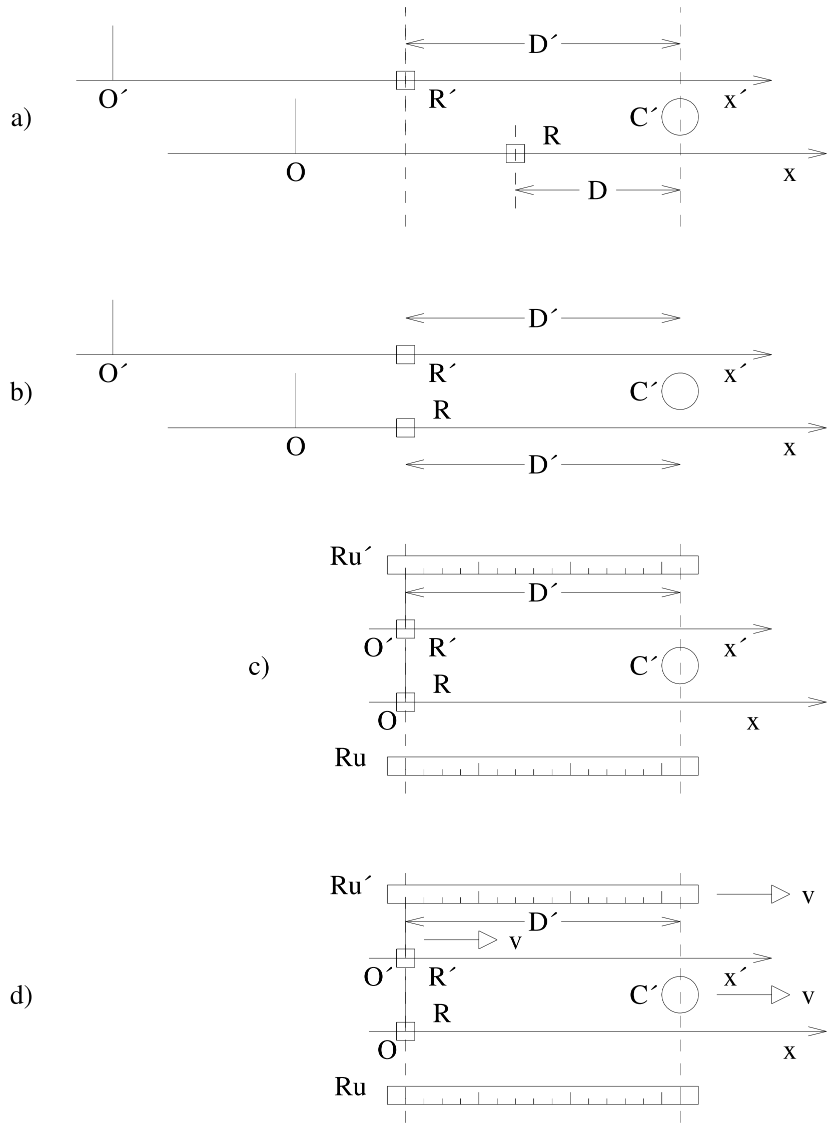

Figure 2: *Spatial coordinate systems and ruler measurements of spatial intervals.* C′*: clock;* R, R′*: reference objects;* Ru, Ru′*: rulers. a) Arbitary cooordinate origins. b) Arbitary cooordinate origins with* R *and* R′ *aligned. c) Coordinate origins,* R *and* R′ *aligned. d) An equal impulsive accleleration has been applied to* C′, R′ *and* Ru′*. See text for discussion.*

by (4.5)-(4.8) on making the replacements:

$$\mathrm{R}' \to \mathrm{C}'_1, \quad \mathrm{C}' \to \mathrm{C}'_2, \quad D'_1 \to 0, \quad D'_2 \to D', \quad t(\mathrm{C}'_1) = t(\mathrm{C}'_2) = \tau' = 0$$

Although the case of equal impulsive accelerations applied to R′, C′ and Ru′ as in Fig. 2d makes particularly clear the invariance of the length interval between R′ and C′, the same conclusion is reached by considering equal acceleration programs, of non-vanishing duration, applied simultaneously to the same three objects. As viewed from the frame S, just after the termination of the acceleration, R′ will be displaced by the same distance, Δx, from the first mark of ruler Ru as C′ is from the sixteenth mark. The interval between C′ and R′ at this epoch is then $(D'+\Delta x)-\Delta x = D'$, the same as when the objects were at rest. As in the common acclerated comoving frame of R′, C′ and Ru′ there is no relative motion of the objects, the separation of C′ and R′ remains D' at all times.

The space and time transformation equations in the primary experiment equivalent to (4.1) and (4.2) can be more conventionally written as:

$$x'(\mathrm{C}') - D' = \gamma(\beta_v)[x(\mathrm{C}') - D' - vt(\mathrm{C})] = 0, \tag{4.10}$$

$$\tau(\mathrm{C}') = \gamma(\beta_v)\left[t(\mathrm{C}) - \frac{\beta_v(x'(\mathrm{C}') - D')}{V}\right]. \tag{4.11}$$

Using the equation of motion: $x(\mathrm{C}')-D' = vt(\mathrm{C}')$ to eliminate $x(\mathrm{C}')-D'$ from (4.11), and taking into account the definition, (3.18), of $\gamma(\beta_v)$, recovers the TD relation (4.2). In fact the time Lorentz transformation equation (4.11) is a necessary consequence of (4.1),(4.2) and the identity $\gamma(\beta_v)^2 - \beta_v^2\gamma(\beta_v)^2 \equiv 1$ and contains the same physical information. The physical content of (4.10) is identical to that of (4.1) and remains the same if the factor $\gamma(\beta_v)$ is replaced by any finite constant. The generic LT derived by Einstein and universally considered in text books and the literature on special relativity corresponds to the particular choice of coordinate systems in S and S′ for which $D' = 0$ in (4.10) and (4.11). In Fig. 2d this implies that O,O′ and C′ are aligned in x when $t(\mathrm{C}') = t(\mathrm{C}) = \tau(\mathrm{C}') = 0$. The error made by Einstein and all subsequent authors, previous to the work presented in Ref. [1], is to assume that (4.10) and (4.11) with $D' = 0$ also correctly describes a synchronised clock in S′ that *does not have* $x' = 0$. On the assumption that (4.10) and (4.11) with $D' = 0$ also describes a synchronised clock $\mathrm{C}'(L')$ at $x' = L'$ it is concluded that:

$$x'(L') = \gamma(\beta_v)[x(L') - vt(\mathrm{C})], \tag{4.12}$$

$$\tau(L') = \gamma(\beta_v)\left[t(\mathrm{C}) - \frac{\beta_v x'(L')}{V}\right], \tag{4.13}$$

where, for example, $x'(L') \equiv x'(\mathrm{C}'(L'))$, while setting $L' = 0$ gives the same equations as (4.10) and (4.11) with $D' = 0$:

$$x'(0) = \gamma(\beta_v)[x(0) - vt(\mathrm{C})], \tag{4.14}$$

$$\tau(0) = \gamma(\beta_v)\left[t(\mathrm{C}) - \frac{\beta_v x'(0)}{V}\right]. \tag{4.15}$$

Subtracting (4.14) from (4.12) then gives the prediction that the spatial separations of $\mathrm{C}'(L')$ and $\mathrm{C}'(0)$ in S are related according to the equation:

$$L' \equiv x'(L') - x'(0) = \gamma(\beta_v)[x(L') - x(0)] \equiv \gamma(\beta_v)L \tag{4.16}$$

which is the LC effect, while subtracting (4.15) from (4.13) gives:

$$\tau(L') - \tau(0) = -\frac{\gamma(\beta_v)\beta_v L'}{V} \tag{4.17}$$

showing that $\mathrm{C}'(L')$ and $\mathrm{C}'(0)$, which are synchronous in S, are not so in the frame S′ which is the RS effect. The flaw in this argument is that the clock $\mathrm{C}'(L')$ described by (4.12) and (4.13) *is synchronised neither with* C′(0) *nor with* C [2, 26, 27, 28]. Such a clock is correctly described by (4.10) and (4.11) with $D' = L'$. For comparison of these equations with (4.12) and (4.13), the former may be written as:

$$x'(L') = \gamma(\beta_v)[x(L') - vt(\mathrm{C})] + X_0, \tag{4.18}$$

$$\tau(L') = \gamma(\beta_v)\left[t(\mathrm{C}) - \frac{\beta_v x'(L')}{V}\right] + T_0, \tag{4.19}$$

where the time-independent additive constants on the right sides of (4.18) and (4.19) are:

$$X_0 = -[\gamma(\beta_v) - 1]L', \tag{4.20}$$

$$T_0 = \frac{\gamma(\beta_v)\beta_v L'}{V}. \tag{4.21}$$

The correct equations (4.18) and (4.19) then differ from the incorrect ones (4.12) and (4.13) by the inclusion of the constants X_0 and T_0. Comparison of (4.17) and (4.21) shows that the spurious RS effect arises from the neglect of the constant T_0 in (4.19), since the addition of T_0 exactly cancels the putative RS effect in (4.17).

The necessity to include such constants to correctly describe synchronised clocks at different spatial positions was clearly stated by Einstein in the original 1905 special relativity paper [5]:

'Macht man über die Anfanslage des bewegten Systems und über den Nullpunkt von τ keinerlei Voraussetzung, so ist auf den rechten Seiten dieser Gleichungen je eine additive Konstante zuzufügen'

or, in English:

'If no assumption whatever be made as to the initial position of the moving system and as to the zero point of τ an additive constant is to be placed on the right side of these equations'

In this statement the symbol τ stands for $t'(\mathrm{C}')$ or $\tau(\mathrm{C}')$ in the notation of the present paper. This important injunction was, however, to the present writer's best knowledge, never respected by Einstein himself, either later in Ref. [5] or in any later work, nor, subsequently, by any other author before the work presented in Ref. [1].

5 The relativistic relative velocity transformation relation

The relativistic relative velocity transformation relation (RRVTR) is derived in a straightforward manner from consideration of the configuration of objects shown in Fig. 2d. Suppose that an object R″ with proper frame S″ moves, like the clock C″ discussed in Section 3 above, with speed u ($> v$) along the positive x-axis in S and is aligned with R and R' at the epoch $t(\mathrm{C}) = 0$. Since its initial separation from C′ is D' and the velocity of R″ relative to C′ in the frame S is $u - v$ it is aligned with C′ at the epoch T in S where

$$T = \frac{D'}{u - v}. \tag{5.1}$$

Denoting the corresponding epoch in S′ by T' and the velocity of R″ in S′ as u' (i.e. u' is the velocity of R″ relative to C′ in S′) then

$$T' = \frac{D'}{u'}. \tag{5.2}$$

Since T and T' are related via the TD relation $T = \gamma(\beta_v)T'$ it follows from (5.1) and (5.2) that the RRVTR is:

$$u' = \gamma(\beta_v)(u - v). \tag{5.3}$$

Setting $u = 0$ gives:

$$u'(u = 0) \equiv -v' = -\gamma(\beta_v)v. \tag{5.4}$$

In the configurations shown in Fig. 1b, v' is the magnitude of the velocity of C relative to C′, as observed in S′ in the primary experiment or the magnitude of the velocity of C′ relative to C, as observed in S in the reciprocal experiment.

Introducing the scaled velocities β_u and $\beta_{u'}$ the RRVTR may be written in three equivalent ways as:

$$\beta_{u'} = \gamma(\beta_v)(\beta_u - \beta_v), \tag{5.5}$$

$$\beta_u = \frac{\beta_{u'}}{\gamma(\beta_v)} + \beta_v, \tag{5.6}$$

$$\beta_v = \frac{\beta_u - \beta_{u'}}{1 + (\beta_{u'})^2}. \tag{5.7}$$

Eq(5.6) is obtained by transposing (5.5), and (5.7) by solving (5.5) for β_v. Inspection of these equations shows that the RRVTR, unlike the transformation equation (3.22) for the parametric velocity in the TD factor, does not constitute a group. Introducing the acronym TDVTR for the Time Dilation Velocity Transformation Relation (3.22) the equations corresponding to (5.5)-(5.7) (identical to them in the Galilean $V \to \infty$ limit) are:

$$\beta_{u'} = \frac{\beta_u - \beta_v}{1 - \beta_u\beta_v}, \tag{5.8}$$

$$\beta_u = \frac{\beta_v + \beta_{u'}}{1 + \beta_v\beta_{u'}}, \tag{5.9}$$

$$\beta_v = \frac{\beta_u - \beta_{u'}}{1 - \beta_u\beta_{u'}}. \tag{5.10}$$

The group operation common to (5.8)-(5.10) is, in words: 'Subtract B from A and then divide this number by one minus B times A'. Comparing instead (5.5) with the inverse transformation (5.6) it can be seen that whereas (5.5) is given by the operation: 'Subtract

B from A and then multiply this number by γ(B)', in (5.6) the order of operations is reversed: 'Divide A by γ(B) and then add B.' Although the operations of addition and subtraction and multiplication and division separately respect the group property, their combined application, in reverse order, in the inverse transformation, does not. Another important difference between the transformations is that the TDVTR has a symmetry property not respected by the RRVTR. The left side of (5.8) is invariant under the exchanges $v \rightarrow -u$, $u \rightarrow -v$, that of (5.9) under $v \rightarrow u'$, $u' \rightarrow v$. Applying the latter exchange to the corresponding RRVTR transformation (5.6) gives $\beta_v/\gamma(\beta_{u'}) + \beta_{u'}$ which is not, in general, equal to $\beta_{u'}/\gamma(\beta_v)+\beta_v$. For example, setting $\beta_v = \sqrt{3}/2$, $\gamma(\beta_v) = 2$ and $\beta_{u'} = \sqrt{15}/4$, $\gamma(\beta_{u'}) = 4$, the former expression gives $\beta_u = 1.35$, the latter $\beta_u = 1.185$.

It is essential at this point to introduce a notation to distinguish certain velocity symbols in the RRVTR and TDVTR that would otherwise be ambiguous (e.g. $\beta_{u'}$ in (5.5) or (5.8)). Relative velocities between inertial frames, used as input parameters to define specific space-time experiments will be denoted by unaccented symbols. Examples are β_u and β_v in (5.5) or (5.8), $\beta_{u'}$ and β_v in (5.6) or (5.9). According to Eq. (3.18), all such scaled 'frame velocities' have an upper limit of unity. Velocities obtained by transformation of frame velocities according to the RRVTR will be denoted by a superposed tilde accent, those transformed by the TDVTR by a superposed caret or circumflex accent. Thus (5.5) and (5.6) are written as:

$$\tilde{\beta}_{u'} = \gamma(\beta_v)(\beta_u - \beta_v), \tag{5.11}$$

$$\tilde{\beta}_u = \frac{\beta_{u'}}{\gamma(\beta_v)} + \beta_v, \tag{5.12}$$

where

$$-1 \le \beta_u,\ \beta_{u'}, \beta_v \le 1, \quad -\infty \le \tilde{\beta}_{u'} \le \infty, \quad -\sqrt{2} \le \tilde{\beta}_u \le \sqrt{2}$$

and (5.8) and (5.9) as:

$$\hat{\beta}_{u'} = \frac{\beta_u - \beta_v}{1 - \beta_u\beta_v}, \tag{5.13}$$

$$\hat{\beta}_u = \frac{\beta_v + \beta_{u'}}{1 + \beta_v\beta_{u'}}, \tag{5.14}$$

where

$$-1 \le \beta_u,\ \beta_{u'}, \beta_v, \hat{\beta}_{u'}, \hat{\beta}_u \le 1.$$

Differentiation of (5.12) with respect to β_v shows that the maximum value, $\sqrt{2}$, of $\tilde{\beta}_u$ occurs when $\beta_{u'} = 1$ and $\beta_v = 1/\sqrt{2}$.

It will be found convenient to use for certain applications, in the interest of clarity, an alternative notation for TD factors and velocities where the moving clock and frame of observation are explicitly indicated. For example: $\gamma(\mathrm{C}') \equiv \gamma(\beta_v)$, $\gamma'(\mathrm{C}'') \equiv \gamma(\beta_{u'})$, $\beta(\mathrm{C}') \equiv \beta_v$, $\beta(\mathrm{C}'') \equiv \beta_u$, $\beta'(\mathrm{C}'') \equiv \beta_{u'}$.

In view of the invariance of transverse length intervals the one-dimensional RRVTR (5.3) generalises for arbitary uniform three-dimensional motion of an object R$''$ to:

$$\tilde{\vec{u'}} = \gamma(\beta_v)(\vec{u} - \vec{v}) \tag{5.15}$$

with the inverse transformation:

$$\vec{\tilde{u}} = \frac{\vec{u}'}{\gamma(\beta_v)} + \vec{v}. \qquad (5.16)$$

Thus to transform a velocity vector $\vec{u}'$ specified in the frame S′ into the frame S, each component is scaled by the appropriate TD factor $\gamma(\beta_v)$:

$$\frac{u'_i}{\gamma(\beta_v)} = \frac{dt'}{dt}\frac{dx'_i}{dt'} = \frac{dx_i}{dt} = (\vec{u}_r)_i, \qquad (5.17)$$

where the invariance of both longitudinal and transverse length intervals $dx'_i = dx_i$, $i = 1,2,3$ has been used. The velocity $\vec{u}_r$ is that of R″ relative to S′ in the frame S. The velocity of R″ in the frame S is then given by classical addition of the $\vec{u}_r$ and $\vec{v}$ in this frame:

$$\vec{\tilde{u}} = \vec{u}_r + \vec{v}. \qquad (5.18)$$

The formulae (5.15) and (5.16) lead to transformations of the angles of velocity three-vectors that differ from those obtained from the three dimensional generalisations of the four vector velocity transformations (3.20) and (3.23). This point will be discussed in a forthcoming article.

The RRVTR (5.11) and its inverse (5.12) are readily generalised to the case of a series of inertial frames in parallel motion. Consider objects A, B, C at rest in the frames S′, S″, S‴ specified by the frame velocities: $\beta(\mathrm{A})$, $\beta'(\mathrm{B})$, $\beta''(\mathrm{C})$. Then (5.11) and (5.12) generalise to

$$\tilde{\beta}''(\mathrm{C}) = \gamma(\hat{\beta}(\mathrm{B}))\left[\beta(\mathrm{C}) - \beta(\mathrm{A}) - \frac{\beta'(\mathrm{B})}{\gamma(\beta(\mathrm{A}))}\right], \qquad (5.19)$$

$$\tilde{\beta}(\mathrm{C}) = \frac{\beta''(\mathrm{C})}{\gamma(\hat{\beta}(\mathrm{B}))} + \frac{\beta'(\mathrm{B})}{\gamma(\beta(\mathrm{A}))} + \beta(\mathrm{A}), \qquad (5.20)$$

$$\hat{\beta}(\mathrm{B}) = \frac{\beta'(\mathrm{B}) + \beta(\mathrm{A})}{1 + \beta'(\mathrm{B})\beta(\mathrm{A})}. \qquad (5.21)$$

Introducing indices i, j to specify frame labels and objects respectively, so that, for example, $\beta(\mathrm{A})$ is written as $\beta(j)^{(i)} = \beta(1)^{(1)}$ or $\beta''(\mathrm{C})$ as $\beta(j)^{(i)} = \beta(3)^{(3)}$ the complete generalisations of (5.11) and (5.12) are:

$$\tilde{\beta}(j)^{(j)} = \gamma(\hat{\beta}(j-1)^{(1)})\left[\beta(j)^{(1)} - \beta(1)^{(1)} - \sum_{l=2}^{j-1}\frac{\beta(l)^{(l)}}{\gamma(\hat{\beta}(l-1)^{(1)})}\right], \qquad (5.22)$$

$$\tilde{\beta}(j)^{(1)} = \sum_{l=2}^{j}\frac{\beta(l)^{(l)}}{\gamma(\hat{\beta}(l-1)^{(1)})} + \beta(1)^{(1)}, \qquad (5.23)$$

where $\tilde{\beta}(1)^{(1)} = \beta(1)^{(1)}$ and $\hat{\beta}(l-1)^{(1)}$ is calculated iteratively according to the TDVTR:

$$\hat{\beta}(l)^{(1)} = \frac{\beta(l)^{(l)} + \hat{\beta}(l-1)^{(1)}}{1 + \beta(l)^{(l)}\hat{\beta}(l-1)^{(1)}}, \qquad (5.24)$$

$$\hat{\beta}(2)^{(1)} = \frac{\beta(2)^{(2)} + \beta(1)^{(1)}}{1 + \beta(2)^{(2)}\beta(1)^{(1)}}. \qquad (5.25)$$

Notice that the TDVTR is employed in (5.24) and (5.25) to calculate the TD factors which appear in the general expressions (5.22) and (5.23) for the RRVTR.

An independent derivation of the inverse RRVTR (5.6) is obtained in the following section from an analysis of velocity transformations in the Hafele-Keating experiment.

6 Space-time and kinematical Lorentz transformations; base and travelling frames and the Hafele-Keating experiment

The space-time Lorentz transformation describing a primary experiment where a synchronised clock C′ at rest in the frame S′ at $x' = D'$ is observed from the frame S may be written in any of the following equivalent forms:

$$x'(\mathrm{C}') = D', \quad x(\mathrm{C}') = D' + vt, \tag{6.1}$$

$$t = \gamma(\beta_v)\tau(\mathrm{C}') \tag{6.2}$$

or

$$\begin{aligned} x'(\mathrm{C}') - D' &= \gamma(\beta_v)[x(\mathrm{C}') - D' - vt] = 0, & (6.3)\\ \tau(\mathrm{C}') &= \gamma(\beta_v)\left[t - \frac{v(x(\mathrm{C}') - D')}{V^2}\right] & (6.4)\end{aligned}$$

or

$$x'(\mathrm{C}') = D', \quad x(\mathrm{C}') - D' = \gamma(\beta_v)[x'(\mathrm{C}') - D' + v\tau(\mathrm{C}')], \tag{6.5}$$

$$t = \gamma(\beta_v)\left[\tau(\mathrm{C}') + \frac{v(x'(\mathrm{C}') - D')}{V^2}\right]. \tag{6.6}$$

The freedom of choice of coordinate origins in S and S' always allows the world lines of C′ in S and S' to be specified in these frames by a single v-independent parameter D', as in (6.1). According to these equations, C′ is synchronised so that $\tau(\mathrm{C}') = t = 0$ when $x(\mathrm{C}') = D'$, for an arbitary value of D'. The epoch t is that registered by a clock similar to C′ (i.e. one that runs at the same rate in its proper frame) that is at rest at an arbitary position in S. Note that the space transformation equation (6.5) is equivalent to Eq. (2.21) and manifests the Inverse Transformation Postulate assumed above to derive the transformation equations. It is obtained by eliminating t between (6.3) and (6.4). The experiment reciprocal to that described by (6.1)-(6.6) is one in which a clock C at rest in the frame S at $x = D$ is observed to move along the negative x-axis in S′ with speed v. The corresponding Lorentz transformation equations are written as[6]:

$$x(\mathrm{C}) = D, \quad x'(\mathrm{C}) = D - vt', \tag{6.7}$$

$$t' = \gamma(\beta_v)\tau(\mathrm{C}) \tag{6.8}$$

[6]Note that (6.1)-(6.6) describe the primary experiment and (6.7)-(6.12) the reciprocal one for any values of the constants D and D', since the choice of spatial coordinate origins in each experiment may be made independently.

or

$$x(\mathrm{C}) - D = \gamma(\beta_v)[x'(\mathrm{C}) - D + vt'] = 0, \quad (6.9)$$

$$\tau(\mathrm{C}) = \gamma(\beta_v)\left[t' + \frac{v(x'(\mathrm{C}) - D)}{V^2}\right] \quad (6.10)$$

or

$$x(\mathrm{C}) = D, \quad x'(\mathrm{C}) - D = \gamma(\beta_v)\left[x(\mathrm{C}) - D - v\tau(\mathrm{C})\right], \quad (6.11)$$

$$t' = \gamma(\beta_v)[\tau(\mathrm{C}) - \frac{v(x(\mathrm{C}) - D)}{V^2}]. \quad (6.12)$$

Here C is synchronised so that so that $\tau(\mathrm{C}) = t' = 0$ when $x'(\mathrm{C}) = D$ for an arbitary value of D. The epoch t' is that registered by a clock similar to C at rest at an arbitary position in S$'$. Notice that, as previously remarked, the transformation equations in the primary experiment contain only the spatial coordinates of the clock C$'$, the position of C being arbitary, whereas, in the reciprocal experiment, only the spatial coordinates of the clock C appear, the position of C$'$ being arbitary. For $D = D'$ the transformation equations (6.3) and (6.9) of the primary and reciprocal experiments, respectively, are identical to the inverse transformations (6.11) and (6.5) of the reciprocal and primary experiments, respectively, when clock labels and symbols distinguishing proper and improper epochs are omitted. This circumstance, in the special case $D' = D = 0$, has in the past caused great confusion concerning the physical interpretation of the transformation equations [31, 32, 33].

The transformation equations (3.20) and (3.23) of the four-vector velocity, U', defined in (3.25) and (3.26) are, using the notation introduced in the previous section:

$$\begin{aligned}
\gamma(\hat{\beta}_u) &\equiv \hat{\gamma}(\mathrm{C}'') = \gamma(\beta_v)[\gamma(\beta_{u'}) + \beta_v\beta_{u'}\gamma(\beta_{u'})] = \gamma(\beta_v)\gamma(\beta_{u'})[1 + \beta_v\beta_{u'}] \\
&= \gamma(\beta_{u'})[\gamma(\beta_v) + \beta_{u'}\beta_v\gamma(\beta_v)] = \hat{\gamma}''(\mathrm{C}), \quad (6.13) \\
\hat{\beta}_u\gamma(\hat{\beta}_u) &\equiv \hat{\beta}(\mathrm{C}'')\hat{\gamma}(\mathrm{C}'') = \gamma(\beta_v)[\beta_{u'}\gamma(\beta_{u'}) + \beta_v\gamma(\beta_{u'})] = \gamma(\beta_v)\gamma(\beta_{u'})[\beta_{u'} + \beta_v] \\
&= \gamma(\beta_{u'})[\beta_v\gamma(\beta_v) + \beta_{u'}\gamma(\beta_{u'})] = \hat{\beta}''(\mathrm{C})\hat{\gamma}''(\mathrm{C}). \quad (6.14)
\end{aligned}$$

The last member of (6.13) describes the transformation of the TD factor, $\gamma'(\mathrm{C}) = \gamma(\beta_v)$, for the clock C, between the frames S$'$ and S, as observed in S$'$, to that, $\hat{\gamma}''(\mathrm{C})$, between the frames S$''$ and S, as observed in S$''$. This experiment is the reciprocal of the one described by the first member of (6.13) and discussed in Section 3 above. Taking the ratio of (6.14) to (6.13) gives

$$\hat{\beta}(\mathrm{C}'') = \frac{\beta_{u'} + \beta_v}{1 + \beta_{u'}\beta_v} = \hat{\beta}''(\mathrm{C}) \quad (6.15)$$

showing, again, the equality of the TD factors in the primary experiment where C$''$ in motion is observed from S: $\hat{\gamma}(\mathrm{C}'') = \gamma(\hat{\beta}(\mathrm{C}''))$, and the reciprocal experiment where C in motion is observed from S$''$: $\hat{\gamma}''(\mathrm{C}) = \gamma(\hat{\beta}''(\mathrm{C}))$. As shown in the derivation of (3.22) and (3.23), each of Eqs. (6.13)-(6.15) is algebraically equivalent to the other two.

The essential equations of the space-time Lorentz transformation are the TD relations (6.2) and (6.8) of the primary and reciprocal experiments. The essential kinematical Lorentz transformation equation is (6.13), which describes how the TD factor γ transforms between two inertial frames. It can be seen that the space-time Lorentz transformation involves only two inertial frames, the first in which a clock is at rest and the velocity of a moving clock is specified, and the second which is the proper frame of the moving clock. The operational definition of the TD factor γ is the ratio of a time interval recorded by the clock at rest to the corresponding interval recorded by the moving clock, as observed from the frame of the stationary clock. It is convenient to introduce a nomenclature for frames in which clocks are at rest, or in motion, and the frame, relative to which, the velocity of the moving clock is specified, in different space-time experiments [7, 34]. In the primary experiment, described by the TD relation (6.2), the rest frame, S, in which the clock C$'$ has frame velocity v is called the *base frame* of the experiment. The rest frame, S$'$ of the moving clock C$'$ in the experiment is called a *travelling frame*. In the reciprocal experiment with the TD relation (6.8), S$'$ is the base frame and S the travelling frame.

Inspection of (6.13), the transformation equation of the TD factor γ, shows that three inertial frames (S, S$'$, S$''$) and three clocks (C, C$'$, C$''$) are necessarily involved in this case. The first member of (6.13) describes the transformation law of the TD factor for the clock C$''$ with S$'$ as base frame and S$''$ as travelling frame to that in an experiment where S is the base frame and S$''$ the travelling frame. The last member of (6.13) (which is merely an algebraic re-arrangement of the first) instead describes the transformation law of the TD factor for the clock C with S$'$ as base frame and S as travelling frame, to that in the experiment with S$''$ as base frame and S as travelling frame. The latter experiment is the reciprocal of the one with S as base frame and S$''$ as travelling frame, and has the same TD factor.

The fundamental relativistic equations (6.2), (6.8) and (6.13) contain only predictions for *the ratios of corresponding time intervals registered by clocks in different inertial frames*. The space-time transformation equations containing spatial coordinates, (6.1) and (6.7), are the same as in Galilean relativity. The more familiar ways as in (6.3), (6.5), (6.9) and (6.11) of writing the space transformations as 'Lorentz transformations' contain however exactly the same physical information as as the simple Galilean equations of motion of the clock at rest in S$'$ or S to be found in (6.1) or (6.7). As discussed in Section 4 above, careful consideration of the choice of spatial coordinate systems and clock synchronisation is essential for the correct physical interpretation of the space transformation equations. When this is done, the spurious nature of the correlated LC and RS effects of conventional special relativity is evident. Since no spatial coordinates appear in the time transformation (TD) equations (6.2) and (6.8), and since only velocities, not space or time coordinates, appear in the kinematical transformation equation (6.13), the choice of spatial coordinates or clock synchronisation constants is irrelevant in physical applications of these transformations.

As discussed in the previous section, three types of velocity symbols with different operational meanings must be introduced to describe, in general, space-time experiments in which clocks in motion are observed. The first are frame velocities, v, specifying the relative velocity of a pair of inertial frames that serves as an input parameter in the definition of a particular space-time experiment. The magnitudes of such velocities have an upper bound of V. The second type are observed velocities $\tilde{v}$ of physical objects as derived from frame velocities by application of the RRVTR. Such velocities, as observed in a travelling frame, may be infinite, but have a definite upper bound (that may be greater than V) in a base frome. Some examples will be discussed below. The third type of velocity, denoted as $\hat{v}$, is the parametric velocity that may be used to specify the TD factor $\gamma(\hat{v}/V)$ that is appropriate to a particular space-time experiment. It is obtained by transforming frame velocities according to the TDVTR, as, for example, in Eq. (6.15). The magnitude of such parametric velocities also has an upper bound of V. A parametric velocity does not, as has mistakenly been assumed hitherto, correspond to any observed velocity of a physical object. It is no more than a convenient parameter to specify the physically-meaningful TD factor γ. All of relativistic kinematics is implicit in the transformation equation (6.13) of γ (or its three-dimensional generalisation) between two inertial frames. Since the scaled parametric velocity is given by the identity: $\hat{\beta} \equiv \sqrt{(\hat{\gamma}^2 - 1)}/\hat{\gamma}$, the space-time physics of uniformly moving clocks may be formulated entirely in terms of the TD factor γ without the necessity to introduce parametric velocities. As will be seen in the following section, the same is true for the concepts of relativistic energy and momentum.

Note that in the simple two-clock experiment described by the TD relations (6.2) and (6.8), where no application of the TDVTR is required, the parametric velocity of the appropriate TD factor is, by Eq. (3.18), equal to the frame velocity of the moving clock in the base frame of the experiment. As will now be discussed, by consideration of some specific examples, the full distinction between v, $\tilde{v}$ and $\hat{v}$ only comes into play in experiments specified by two initial frame velocites in which three clocks, at rest in different inertial frames, are involved. The Hafele-Keating experiment, to be described below, was an actual realisation of such an experiment.

Considering first a space-time experiment involving only two clocks, one at rest and the other in motion, it is clear that observers at rest relative to the clocks know at once whether they are in the travelling frame or in the base frame. If they are in the base frame the moving clock is seen to run slower than a clock at rest. If they are in a travelling frame, it will be seen to run faster. Such an experiment seems, at first sight, to have been considered by Einstein in the 1905 special relativity paper [5], where it is stated, in reference to clocks on the surface of the Earth:

'...a balance-clock at the equator . . .must go more slowly, by a small amount, than a precisely similar clock situated at the poles under otherwise identical conditions.'

Einstein was referring here only to the special-relativistic TD effect. As was was pointed out some sixty years later [35], when the general-relativistic effect of the Earth's gravity is taken into account, all clocks on the Earth's surface 'go' at the same speed, as the special-relativistic 'red-shift' and the gravitational 'blue-shift'of the clock frequency exactly cancel. Einstein didn't calculate the size of the TD effect for an observer situated at the pole, but one may guess it would have been assumed that the appropriate value of v in the TD factor was the velocity of the equatorial clock relative to the Earth's pole, that is, $v = \Omega R$ where Ω is the angular velocity of the Earth and R its equatorial radius. This would have been the correct assumption. Consider now, however, observation of the equatorial clock from a position on the Earth's surface with latitude angle θ. Since, neglecting corrections of $O(\beta^2)$, the relative velocity of this observer and the equatorial clock is $\Omega R(1 - \cos\theta)$ it might, naively, be expected that this is the appropriate value of v to insert in the TD factor $\gamma(v)$. If this were indeed the case, then the exact cancellation of special- and general-relativistic effects mentioned above would not occur, and the results of the Hafele-Keating experiment, to be described below, would have been completely different to the ones actually obtained. In fact, clocks at rest at any position on the

Earth's surface, except at the poles, are in travelling frames with different velocities, v, determined by Ω, and their perpendicular distance $R_{\perp}$ from the Earth's axis of rotation, relative to an observer situated at one of the poles of the Earth. The North and South Poles of the Earth are at rest in a base frame co-moving with the centroid of the Earth, for example, the Earth Centered Inertial (ECI) frame with constant orientation relative to the fixed stars used for clock synchronisation by the Global Positioning System (GPS) satellites [36]. Calling the latter frame S, the equatorial clock C$''$, and a clock at latitude θ, C$'$, for consistency with previous notation, the TD relations for an observer in S with local clock C are:

$$t(\mathrm{C}) = \gamma(\beta(0))\tau(\mathrm{C}''), \tag{6.16}$$

$$t(\mathrm{C}) = \gamma(\beta(\theta))\tau(\mathrm{C}'), \tag{6.17}$$

where

$$\beta(\theta) = \frac{\Omega R \cos\theta}{V} \tag{6.18}$$

and, for simplicity, the Earth has been taken to be spherical. The TD factor $\bar{\gamma}'(\theta)$ for observation of C$''$ from the position of C$'$ is given by (6.16) and (6.17) as[7]:

$$t'(\mathrm{C}') = \tau(\mathrm{C}') = \bar{\gamma}'(\theta)\tau(\mathrm{C}'') \equiv \frac{\gamma(\beta(0))\tau(\mathrm{C}'')}{\gamma(\beta(\theta))}. \tag{6.19}$$

Since $\Omega R \simeq 0.47$km/sec and, anticipating the result to be found below that $V = c = 3 \times 10^5$km/sec, then $\beta(\theta) \ll 1$, and to order β^2,

$$\bar{\gamma}'(\theta) = 1 + \frac{1}{2}\left(\frac{\Omega R}{V}\right)^2 \sin^2\theta. \tag{6.20}$$

If, instead, the relative velocity of C$''$ and C$'$ is used to calculate the TD factor at the same level of approximation, the factor $\sin^2\theta$ in (6.20) is replaced by $(1-\cos\theta)^2$. For $\theta = 45^\circ$ the β^2 term in (6.20) is a factor of 5.8 greater than that given by using the velocity of C$''$ relative to C$'$ to calculate γ'. The distinction in the above calculation between the base frame S and the travelling frames S$'$ and S$''$ is clearly of crucial importance for the correct analysis of the problem. The base frame time t(C) in (6.16) and (6.17) has previously been called in the literature 'coordinate time' when using the ECI frame to calculate time dilation corrections, using a formula similar to (6.20), for clocks on GPS satellites as viewed from the surface of the Earth [36], or when analysing the Hafele-Keating experiment [37].

Consideration of the above example sheds light on the operational meaning of the transformation formulae for the TD parametric velocity (5.13) and (5.14) above. An application of (5.13) to clocks at different positions on the surface of the Earth using the ECI frame as base gives:

$$\hat{\beta}'(\mathrm{C}'') = \frac{\beta(0) - \beta(\theta)}{1 - \beta(0)\beta(\theta)}, \tag{6.21}$$

or to order β^2:

$$\gamma'(\theta) = \gamma(\hat{\beta}'(\mathrm{C}'')) = 1 + \frac{1}{2}\left(\frac{\Omega R}{V}\right)^2 (1-\cos\theta)^2 \tag{6.22}$$

[7]The bar accent is introduced to distinguish the appropriate TD factor here for the clock C$''$ from the naive one calculated from the relative velocity of the observer and the clock. See the discussion of Eq. (6.22) below.

instead of the correct formula (6.20). This means that the transformation formula, (5.13), where both frame velocities are specified in the base frame, unlike that, (5.14), where one frame velocity is specified in the base frame and the other in the travelling frame, should not be interpreted as a calculation of TD factor in the space-time experiment considered, but rather as a kinematical transformation relating the base frame configuration of one experiment to the base frame configuration of another, physically independent, one. A simple example of this is given by setting $\beta_u = 0$ in (5.13) so that $\hat{\beta}'_{u'} = -\beta_v$. In this case the independent experiment is simply the reciprocal of the original one in which S is the base frame and S′ the travelling frame, i.e. that in which S′ is the base frame and S the travelling frame.

In the example just considered, the input frame velocity parameter $\beta(\theta)$ specifying the travelling frame is calculated in the base frame by considering the spatial geometry of the surface of the rotating Earth. The clocks C′ and C″ are actually in a rotating, and therefore transversely-accelerated frame. The corresponding inertial frames S′ and S″ are therefore actually the instantaneous comoving ones at any instant. It was, however, demonstrated experimentally by precise measurement of the TD effect for decaying muons in near-circular orbits in a storage ring at CERN [38] that there is no temporal effect of such transverse acceleration. The observed TD effect in the experiment was the same as that predicted for rectilinear motion with the same velocity, in spite of a transverse acceleration of 10^{18}g.

In the case where the frame velocities of both clocks are specified in the base frame, the appropriate RRVTR is (5.11). The prediction of this formula is illustrated for $\beta_u \equiv \beta(\mathrm{C}'') = 0.3$ and various values of $\beta_v \equiv \beta(\mathrm{C}')$ in Fig. 3. The observed velocity in S′, $\tilde{\beta}_{u'} \equiv \tilde{\beta}'(\mathrm{C}'')$ decreases monotonically tending to minus infinity as $\beta(\mathrm{C}') \rightarrow 1$. Also shown in this figure is the $\beta(\mathrm{C}')$ dependence of the TD effect for C″, as observed from S′, calculated according to (6.19) as $\bar{\gamma}'(\mathrm{C}'') = \gamma(\mathrm{C}'')/\gamma(\mathrm{C}')$ in comparison with $\gamma'(\tilde{\beta}'(\mathrm{C}''))$, i.e. using, incorrectly, the observed velocity of C″ in S′ as a parametric velocity to calculate the TD effect. Since $|\tilde{\beta}'(\mathrm{C}'')| > 1$ for large values of $\beta(\mathrm{C}')$, $\gamma'(\tilde{\beta}'(\mathrm{C}''))$ is unphysical (imaginary) in this region. Also, since $\bar{\gamma}'(\mathrm{C}') < 1$ for $\beta(\mathrm{C}') > 0.3$ this TD factor cannot be parameterised by any real value of $\hat{\beta}'$.

A second type of space-time experiment involving one stationary and two moving clocks can be considered, This is an experiment in which one travelling frame velocity (say $\beta(\mathrm{C}')$) is specified in the base frame and the other (say $\beta'(\mathrm{C}'')$) in the first travelling frame, instead of the base frame. In this case $\hat{\gamma}(\mathrm{C}'') = \gamma(\hat{\beta}(\mathrm{C}''))$ is calculated according to Eq. (6.13) or, alternatively, acccording to Eq. (6.15) where $\beta_v = \beta(\mathrm{C}')$ and $\beta_{u'} = \beta'(\mathrm{C}'')$, while the observed velocity of C″ in S is given by (5.12) where $\beta_{u'} = \beta'(\mathrm{C}'')$. Some predictions of (5.12) for $\tilde{\beta}_u = \tilde{\beta}(\mathrm{C}'')$, (6.13) for $\hat{\gamma}(\mathrm{C}'')$ and of (3.18) for $\gamma(\mathrm{C}')$, all as a function of $\beta(\mathrm{C}')$, for various fixed values $\beta'(\mathrm{C}'')$, are shown in Figs. 4-7. Also shown are predictions for the TD effect, as observed in the travelling frame S′, $\bar{\gamma}'(\mathrm{C}'')$, calculated according to Eq. (6.19) as well as the 'naive' prediction: $\gamma'(\beta'(\mathrm{C}''))$ given by setting $\hat{\beta}' = \beta'(\mathrm{C}'')$ in the TD factor $\gamma' = \gamma(\hat{\beta}')$.

Some general features of the predictions shown in Figs.4-7 are:

a) For all values of $\beta'(\mathrm{C}'')$, the gradient of $\hat{\gamma}(\mathrm{C}'')$ is a monotonically increasing function of $\beta(\mathrm{C}')$. In general the corresponding parametric velocity $\hat{\beta}(\mathrm{C}'')$ is not equal to the

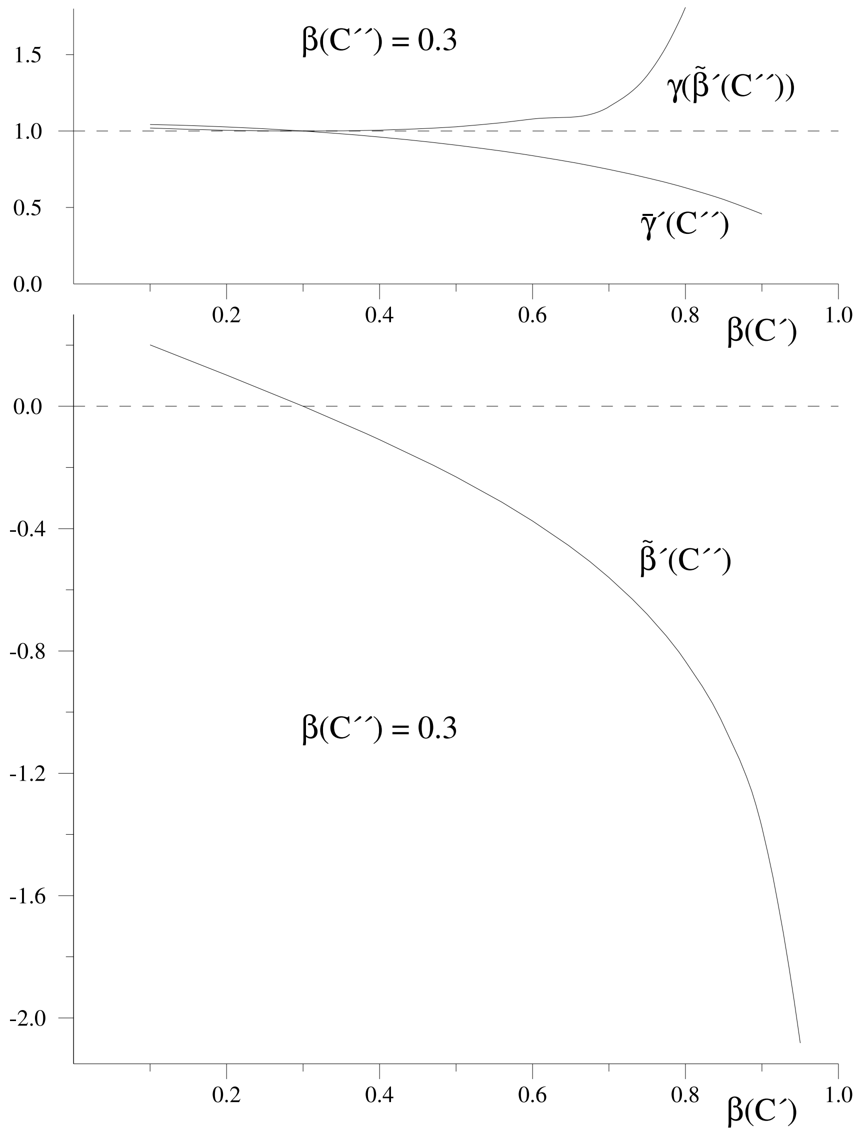

Figure 3: *Time dilation factors and observed velocity of* C'' *as a function of* $\beta(\mathrm{C}')$. $\tilde{\beta}'(\mathrm{C}'')$ *is calculated with the RRVTR (5.11).* $\bar{\gamma}'(\mathrm{C}'') \equiv \hat{\gamma}(\mathrm{C}'')/\gamma(\mathrm{C}')$, *where* $\hat{\gamma}'(\mathrm{C}'')$ *is calculated with (6.13) and* $\gamma(\mathrm{C}')$ *with (3.18).* $\beta(\mathrm{C}'') = 0.3$.

observed velocity $\tilde{\beta}(\mathrm{C}'')$. For any given value of $\beta'(\mathrm{C}'')$ there is, however, one value of $\beta(\mathrm{C}')$ for which $\hat{\beta}(\mathrm{C}'') = \tilde{\beta}(\mathrm{C}'')$, so that the observed TD effect in the frame S is the same as if $\tilde{\beta}(\mathrm{C}'')$ was actually the frame velocity of C″ in S. Setting $\tilde{\beta}_u = \hat{\beta}_u$ and using (5.12) and (5.14) to find the value of β_v corresponding to a fixed value of $\beta_{u'} = \beta'(\mathrm{C}'')$, it is found that

$$\beta_v = \beta(\mathrm{C}') = -\frac{2\beta'(\mathrm{C}'')}{1+\beta'(\mathrm{C}'')^2}, \tag{6.23}$$

and

$$\hat{\beta}(\mathrm{C}'') = \tilde{\beta}(\mathrm{C}'') = -\beta'(\mathrm{C}''). \tag{6.24}$$

b) For positive (negative) values of $\beta'(\mathrm{C}'')$, $\bar{\gamma}'(\mathrm{C}'')$ is a monotonically increasing (decreasing) function of $\beta(\mathrm{C}')$.

c) For positive values of $\beta'(\mathrm{C}'')$, the observed value, $\tilde{\beta}(\mathrm{C}'')$, of C″ in S is equal to $\beta'(\mathrm{C}'')$ when $\beta(\mathrm{C}') = 0$, has a maximum value, and then tends to unity as $\beta(\mathrm{C}') \to 1$.

Differentiating (5.12) with respect to $\beta_v = \beta(\mathrm{C}')$ gives:

$$\frac{d\tilde{\beta}(\mathrm{C}'')}{d\beta(\mathrm{C}')} = 1 - \gamma(\beta(\mathrm{C}'))\beta(\mathrm{C}')\beta'(\mathrm{C}''). \tag{6.25}$$

The derivative tends to minus infinity as $\beta'(\mathrm{C}') \to 1$ for all positive values of $\beta'(\mathrm{C}'')$. The maximum value of $\tilde{\beta}(\mathrm{C}'')$:

$$\tilde{\beta}(\mathrm{C}'')_{\mathrm{Max}} = \frac{1}{\sqrt{1+\beta'(\mathrm{C}'')^2}}\left[\beta'(\mathrm{C}'')^2 + \frac{1}{\beta'(\mathrm{C}'')}\right] \tag{6.26}$$

occurs for

$$\beta(\mathrm{C}')_{\mathrm{Max}} = \frac{1}{\beta'(\mathrm{C}'')\sqrt{1+\beta'(\mathrm{C}'')^2}}. \tag{6.27}$$

The absolute maximum value of $\tilde{\beta}(\mathrm{C}'')$ of $\sqrt{2}$ corresponds to $\beta'(\mathrm{C}'') = 1$ and $\beta(\mathrm{C}')_{\mathrm{Max}} = 1/\sqrt{2}$.

For negative values of $\beta'(\mathrm{C}'')$, $\tilde{\beta}(\mathrm{C}'')$ increases monotonically with $\beta'(\mathrm{C}')$, tending to unity with infinite gradient, as $\beta'(\mathrm{C}') \to 1$.

The predictions shown in Figs. 4 and 5 or 6 and 7 that $\bar{\gamma}'(\mathrm{C}'')$ is greater than unity when $\beta'(\mathrm{C}'')$ is positive and less than unity when $\beta'(\mathrm{C}'')$ is negative has been experimentally verified in an experiment performed by Hafele and Keating (HK) [15]. In this experiment, performed in 1971, four caesium beam atomic clocks were flown around the world in commercial aircraft once from west to east (W−E) and once from east to west (E−W). After the round trips, the clocks were compared with Earth-bound reference clocks at the U.S. Naval Observatory. Differences between the time intervals recorded by the airborne and Earth-bound clocks arise from both special relativistic and general relativistic (gravitational) effects. The gravitational blue-shift resulting from the Earth's gravitational potential, which increases with altitude[8], causes the airborne clocks to run faster as observed from the surface of the Earth. Here only the special relativistic effects in the experiment will be considered. In correspondence with the above discussion of clocks on the surface of the Earth, the following inertial frames are defined:

[8]The potential is negative at finite amplitudes and vanishes, by definition, at infinite altitude.

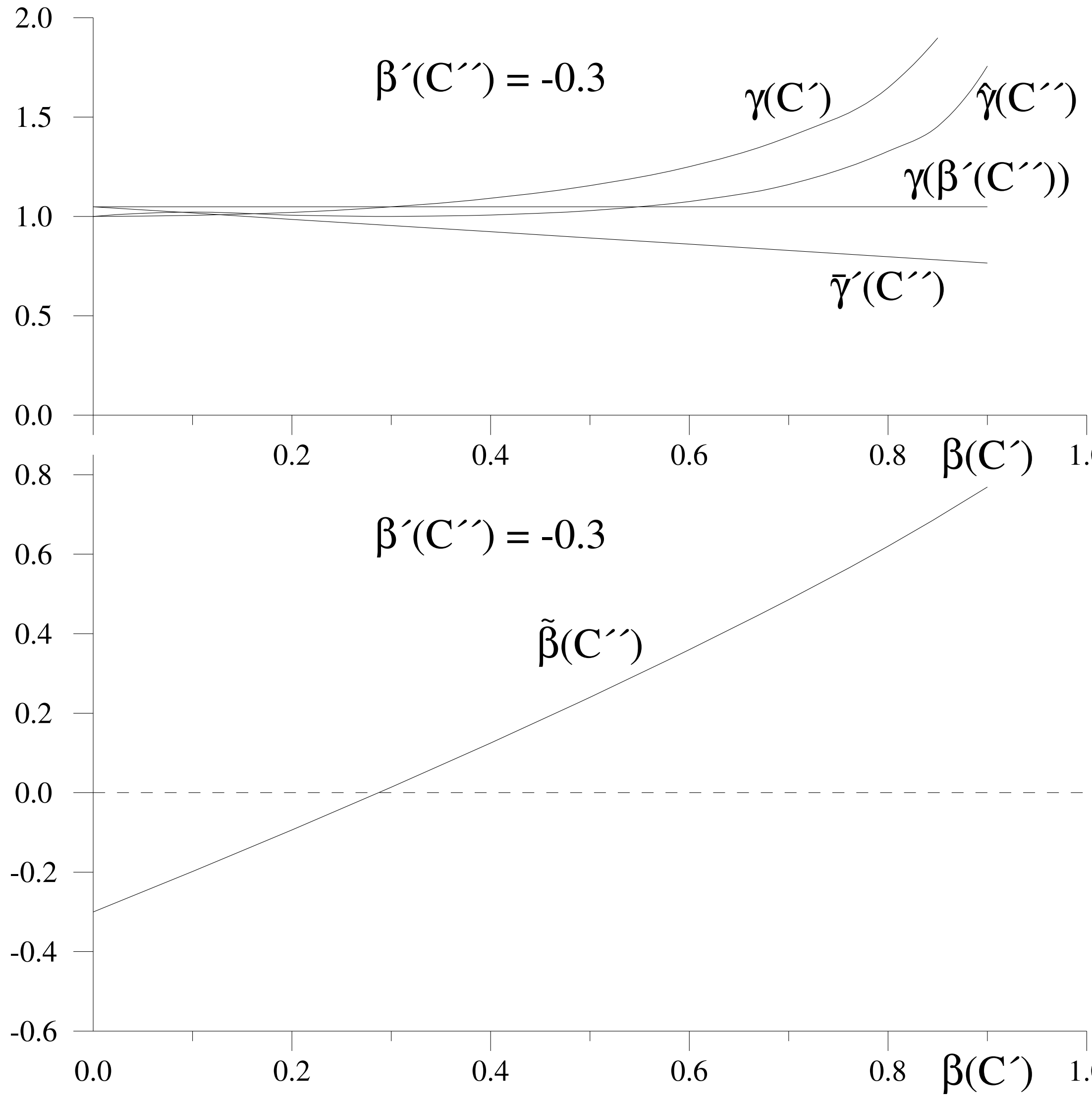

Figure 4: *Time dilation factors and observed velocity of* C$''$ *as a function of* $\beta(\mathrm{C}')$. $\tilde{\beta}(\mathrm{C}'')$ *is calculated with the RRVTR (5.12),* $\hat{\gamma}(\mathrm{C}'')$ *with (6.13) and* $\gamma(\mathrm{C}')$ *with (3.18).* $\bar{\gamma}'(\mathrm{C}'') \equiv \hat{\gamma}(\mathrm{C}'')/\gamma(\mathrm{C}')$. $\beta'(\mathrm{C}'') = -0.3$.

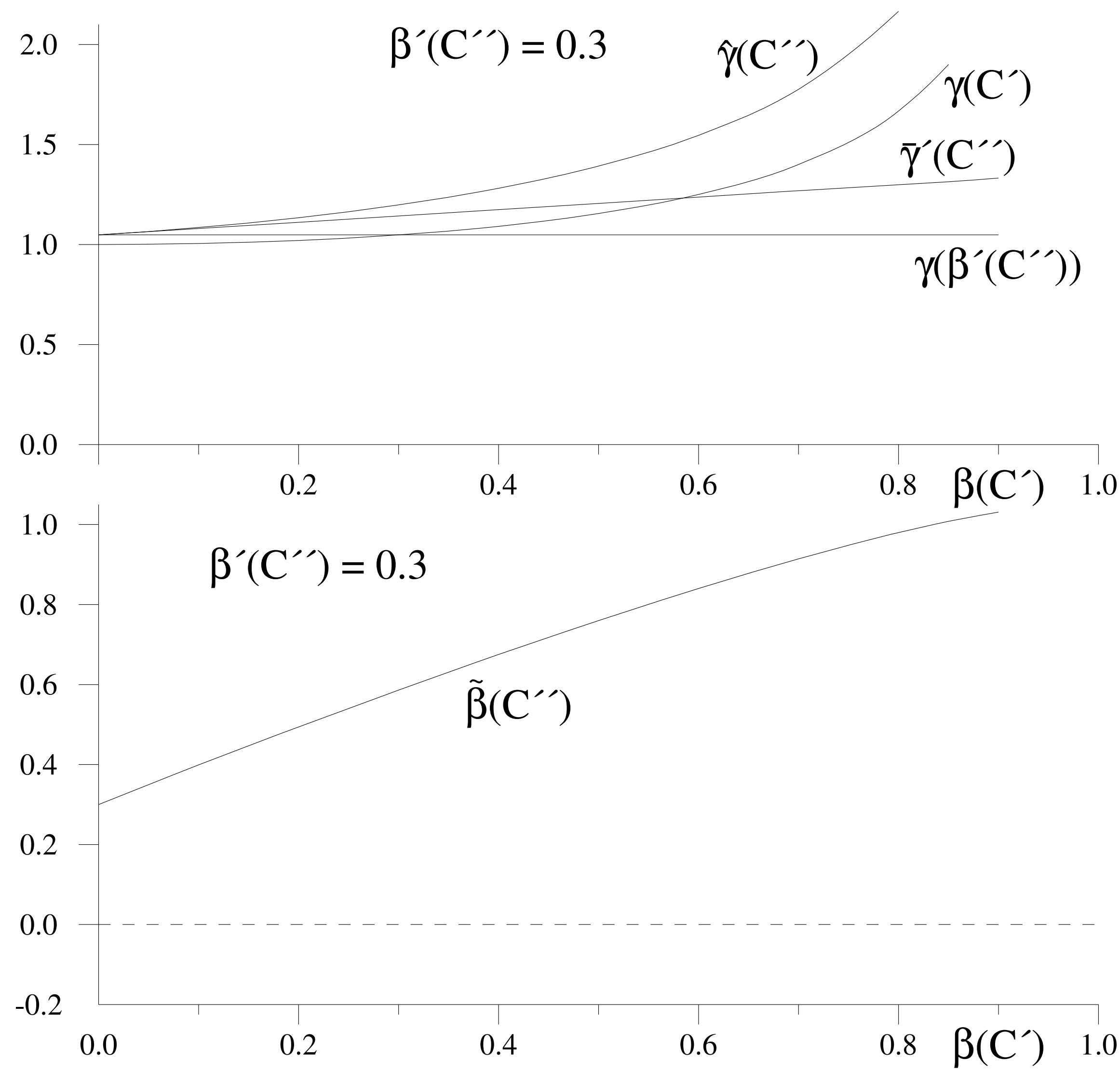

Figure 5: *Time dilation factors and observed velocity of* C″ *as a function of* $\beta(\mathrm{C}')$. $\tilde{\beta}(\mathrm{C}'')$ *is calculated with the RRVTR (5.12),* $\hat{\gamma}(\mathrm{C}'')$ *with (6.13) and* $\gamma(\mathrm{C}')$ *with (3.18).* $\bar{\gamma}'(\mathrm{C}'') \equiv \hat{\gamma}(\mathrm{C}'')/\gamma(\mathrm{C}')$. $\beta'(\mathrm{C}'') = 0.3$.

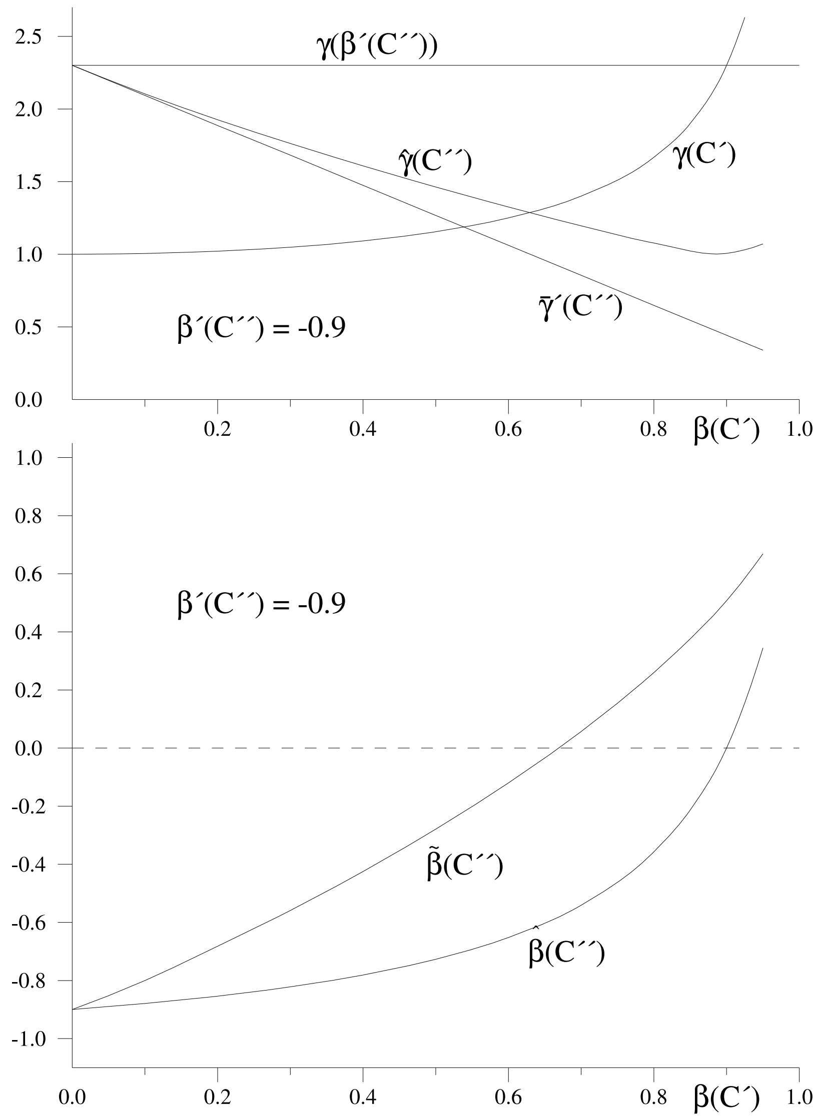

Figure 6: *Time dilation factors and observed velocity of* C'' *as a function of* $\beta(\mathrm{C}')$. $\tilde{\beta}(\mathrm{C}'')$ *is calculated with the RRVTR (5.12),* $\hat{\gamma}(\mathrm{C}'')$ *with (6.13) and* $\gamma(\mathrm{C}')$ *with (3.18).* $\bar{\gamma}'(\mathrm{C}'') \equiv \hat{\gamma}(\mathrm{C}'')/\gamma(\mathrm{C}')$. $\beta'(\mathrm{C}'') = -0.9$.

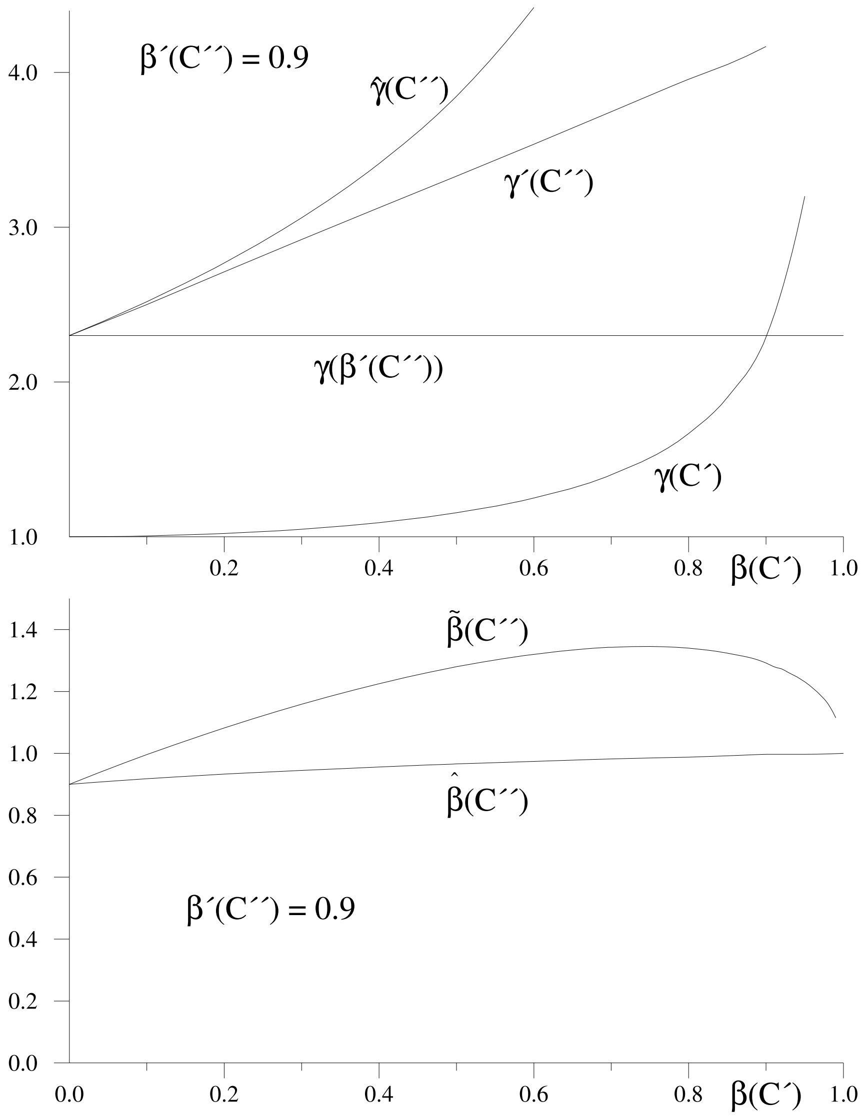

Figure 7: *Time dilation factors and observed velocity of* C'' *as a function of* $\beta(\mathrm{C}')$. $\tilde{\beta}(\mathrm{C}'')$ *is calculated with the RRVTR (5.12),* $\hat{\gamma}(\mathrm{C}'')$ *with (6.13) and* $\gamma(\mathrm{C}')$ *with (3.18).* $\bar{\gamma}'(\mathrm{C}'') \equiv \hat{\gamma}(\mathrm{C}'')/\gamma(\mathrm{C}')$. $\beta'(\mathrm{C}'') = 0.9$.

S : Comoving (non-rotating) frame of the centroid of the Earth. The rotation of the latter about the Sun is neglected.

S$'$: Instantaneous comoving frame of clock at rest on the surface of the Earth at the equator.

S$''$: Instantaneous comoving frame of clock at rest in the aircraft.

As discussed above, although the clocks at rest in the frames S$'$ and S$''$ actually undergo transverse acceleration, in virtue of their circular motion, the TD effect is the same as that in an inertial frame with the same constant velocity. The TD effects between the base frame S and the travelling frames S$'$ and S$''$ are then described by equations similar to (6.16) and (6.18):

$$t(\mathrm{C}) = \gamma(\beta(\mathrm{C}'))\tau(\mathrm{C}') \equiv \gamma(\mathrm{C}')\tau(\mathrm{C}'), \tag{6.28}$$
$$t(\mathrm{C}) = \gamma(\hat{\beta}(\mathrm{C}''))\tau(\mathrm{C}'') \equiv \hat{\gamma}(\mathrm{C}'')\tau(\mathrm{C}'') \tag{6.29}$$

so that, similar to (6.19):

$$t'(\mathrm{C}') = \tau(\mathrm{C}') = \bar{\gamma}'_{\mathrm{obs}}\tau(\mathrm{C}'') = \frac{\hat{\gamma}(\mathrm{C}'')}{\gamma(\mathrm{C}')}\tau(\mathrm{C}''). \tag{6.30}$$

The transformation equation (6.13) of the TD factor $\gamma(\beta_{u'}) \equiv \gamma'(\mathrm{C}'')$ and (6.30) then give:

$$\bar{\gamma}'_{\mathrm{obs}} = \frac{\hat{\gamma}(\mathrm{C}'')}{\gamma(\mathrm{C}')} = \gamma'(\mathrm{C}'')[1 + \beta(\mathrm{C}')\beta'(\mathrm{C}'')]. \tag{6.31}$$

Considering equatorial circumnavigation of the Earth by an aircraft moving with constant speed v'_{A} relative to the surface of the Earth, the TD factors for an observer at rest on the surface of the Earth for the W−E and E−W circumnavigations are:

$$\bar{\gamma}'(\mathrm{W}-\mathrm{E})_{\mathrm{obs}} = \gamma(\beta'_{\mathrm{A}})(1 + \beta'_{\mathrm{A}}\beta_{\mathrm{E}}), \tag{6.32}$$
$$\bar{\gamma}'(\mathrm{E}-\mathrm{W})_{\mathrm{obs}} = \gamma(\beta'_{\mathrm{A}})(1 - \beta'_{\mathrm{A}}\beta_{\mathrm{E}}) \tag{6.33}$$

where

$$\beta'_{\mathrm{A}} \equiv \frac{v'_{\mathrm{A}}}{V}, \tag{6.34}$$
$$\beta_{\mathrm{E}} \equiv \frac{\Omega R}{V}. \tag{6.35}$$

If the time intervals, recorded during circumnavigation, by the Earth-bound clock C$'$ and the airborne clock C$''$ are T' and T'' respectively then:

$$\Delta T' \equiv T'' - T' = T'\left[\frac{1}{\bar{\gamma}'_{\mathrm{obs}}} - 1\right] = T'\left[\frac{1}{\gamma(\beta'_{\mathrm{A}})(1 \pm \beta'_{\mathrm{A}}\beta_{\mathrm{E}})} - 1\right]. \tag{6.36}$$

with $+\beta'_{\mathrm{A}}$ ($-\beta'_{\mathrm{A}}$) corresponding to the W−E (E−W) flights. Retaining only the order β^2 terms in the expansion of $1/\bar{\gamma}'_{\mathrm{obs}} - 1$ gives:

$$\Delta T'(\mathrm{W}-\mathrm{E}) = -\frac{T'\beta'_{\mathrm{A}}}{2}(\beta'_{\mathrm{A}} + 2\beta_{\mathrm{E}}), \tag{6.37}$$
$$\Delta T'(\mathrm{E}-\mathrm{W}) = \frac{T'\beta'_{\mathrm{A}}}{2}(-\beta'_{\mathrm{A}} + 2\beta_{\mathrm{E}}). \tag{6.38}$$

With $V = c = 3 \times 10^8$ m/s, $v'_{\rm A} = 300$ m/s and $v_{\rm E} = \Omega R = 470$ m/s corresponding to a round-trip time of $T' = 27$ h, (6.37) and (6.38) give:

$$\Delta T'({\rm W-E}) = -201 \text{ ns},$$

$$\Delta T'({\rm E-W}) = 104 \text{ ns}$$

to be compared with the predictions for the special relativistic (SR) contributions to $\Delta T'$ for the actual flight paths followed in the HK experiment [15]:

$$\Delta T'_{\rm HK}({\rm W-E})_{\rm SR} = -184 \pm 18 \text{ ns},$$

$$\Delta T'_{\rm HK}({\rm E-W})_{\rm SR} = 90 \pm 10 \text{ ns}.$$

Including also the general relativistic 'time contraction' effects gave the overall predictions [15]:

$$\Delta T'_{\rm HK}({\rm W-E})_{\rm SR+GR} = -40 \pm 23 \text{ ns},$$

$$\Delta T'_{\rm HK}({\rm E-W})_{\rm SR+GR} = 275 \pm 21 \text{ ns}$$

in good agreement with the experimentally measured values:

$$\Delta T'_{\rm HK}({\rm W-E})_{\rm meas} = -59 \pm 10 \text{ ns},$$

$$\Delta T'_{\rm HK}({\rm E-W})_{\rm meas} = 273 \pm 7 \text{ ns}.$$

In Hafele's original calculation of the expected time differences [39] a formula somewhat different to (6.36) was used:

$$\Delta T' = T' \left[\frac{\gamma(\beta_{\rm E})}{\gamma([\beta_{\rm E} \pm \beta'_{\rm A}])} - 1 \right] \tag{6.39}$$

with $+\beta'_{\rm A}$ for the W−E and $-\beta'_{\rm A}$ for the E−W circumnavigations. This equation assumes the classical relative velocity transformation $\tilde{\beta}_{\rm A} = \beta_{\rm E} \pm \beta'_{\rm A}$ between the frames S and S′ instead of Eq. (5.6), and makes no distinction between $\tilde{\beta}_{\rm A}$ and $\hat{\beta}_{\rm A}$. However, to order β^2, (6.36) and (6.39) give the same predictions (6.37) and (6.38). In the HK experiment where $\beta^2 \simeq 10^{-12}$, the order β^4 corrections distinguishing (6.36) or (6.39) from (6.37) and (6.38) are completely neligible.

The correct choice of base and travelling frames for the calculation of the TD effects is essential in the analysis of the HK experiment presented above and implicit in (6.39). Choosing S′ instead of S as the base frame then $\bar{\gamma}'_{\rm obs}$ in (6.36) is replaced by $\gamma(\beta'_{\rm A})$ predicting equal values of $\Delta T'({\rm W-E})$ and $\Delta T'({\rm E-W})$ of -49 ns, while choosing S″ as base frame and S′ as travelling frame (the reciprocal experiment) gives:

$$\Delta T'({\rm W-E}) = \Delta T'({\rm E-W}) = 49 \text{ ns}$$

in contradiction with the different special relativistic contributions on the W−E and E−W flights measured in the experiment.

In Ref. [39] Hafele makes the incorrect remark that the time interval difference predictions of the clocks, (6.37) and (6.38), are not the same as the naive prediction, that the TD effect depends only on the relative velocity of the two observed clocks, due to the effect of the transverse accelerations experienced by both the airborne and Earth-bound clocks. As pointed out above, and as experimentally verified [38], tranversely accelerated clocks with velocities of constant magnitude show the same TD effect as clocks in rectilinear motion with the same velocity.

The HK experiment also demonstrates, in a clear manner, the different physical meanings of the RRVTR and the TDVTR. The former describes the observed relative velocities in different frames in the same space-time experiment, the latter, transformations of the parametric velocity that may (but does not need to be) used to specify, according to Eqs. (3.18) and (3.19), the TD factor γ in a space-time experiment involving at least three inertial frames, as is the case for the HK experiment.

The distances $d(\mathrm{W-E})$, $d(\mathrm{E-W})$ travelled by the aircraft in the frame S during their circumnavigations are[9]:

$$d(\mathrm{W-E}) = v_{\mathrm{E}}T(\mathrm{C}) + 2\pi R, \tag{6.40}$$

$$d(\mathrm{E-W}) = v_{\mathrm{E}}T(\mathrm{C}) - 2\pi R, \tag{6.41}$$

where $T(\mathrm{C})$ is the flight time, in either direction, as observed in the frame S. The corresponding flight time as observed in the frame S' is:

$$T'(\mathrm{C}') = \frac{2\pi R}{v'_{\mathrm{A}}}, \tag{6.42}$$

where

$$T(\mathrm{C}) = \gamma(\mathrm{C}')T'(\mathrm{C}'). \tag{6.43}$$

Notice here the important point that, since the speed of the aircraft relative to the surface of the Earth is the same for the W−E and E−W flights, both flights take the same time, $T'(\mathrm{C}')$, as recorded by the Earth-bound clock, and hence, from (6.43), also as recorded by the clock C. It follows from (6.40) and (6.41) that

$$v_{\mathrm{A}}(\mathrm{W-E}) \equiv \frac{d(\mathrm{W-E})}{T(\mathrm{C})} = v_{\mathrm{E}} + \frac{v'_{\mathrm{A}}}{\gamma(\mathrm{C}')}, \tag{6.44}$$

$$v_{\mathrm{A}}(\mathrm{E-W}) \equiv \frac{d(\mathrm{E-W})}{T(\mathrm{C})} = v_{\mathrm{E}} - \frac{v'_{\mathrm{A}}}{\gamma(\mathrm{C}')}. \tag{6.45}$$

These are examples of the inverse RRVTR as in Eq.(5.6) above.

Consider now the distance, $\Delta s'(\mathrm{W-E})$, of the Eastward flying aircraft from the ground station in the frame S′ [10], after a time interval $\Delta t'(\mathrm{C}')$, sufficiently short that the curvature of the surface of the Earth may be neglected. The distance moved by the aircraft during the corresponding time interval $\Delta t(\mathrm{C})$ in the frame S is:

$$\Delta d(\mathrm{W-E}) = v_{\mathrm{A}}(\mathrm{W-E})\Delta t(\mathrm{C}) = v_{\mathrm{E}}\Delta t(\mathrm{C}) + \Delta s(\mathrm{W-E}), \tag{6.46}$$

[9] I thank Brian Coleman for sending me an unpublished paper in which the base frame path equations (6.40) and (6.41), which I had not hitherto considered, were written down.

[10] As previously, the distance of the aircraft above the surface of the Earth is neglected in the calculation of special relativistic effects.

where $\Delta s(\mathrm{W-E})$ is the separation of the aircaft and the ground station in the frame S, after the time interval $\Delta t(\mathrm{C})$. Combining (6.44) and (6.46) gives:

$$\Delta s(\mathrm{W-E}) = (v_{\mathrm{A}}(\mathrm{W-E}) - v_{\mathrm{E}})\Delta t(\mathrm{C}) = \frac{v'_{\mathrm{A}}\Delta t(\mathrm{C})}{\gamma(\mathrm{C}')} = v'_{\mathrm{A}}\Delta t'(\mathrm{C}') = \Delta s'(\mathrm{W-E}), \quad (6.47)$$

where the TD relation $\Delta t(\mathrm{C}) = \gamma(\mathrm{C}')\Delta t'(\mathrm{C}')$ has been used. This demonstrates the invariance of corresponding length intervals on the surface of the Earth (comoving frame S′) and in the base frame S, without the necessity to consider specific spatial coordinate systems in these frames. This is consistent with the conclusion of Section 4 above concerning the absence of any LC effect. The RRVTR was derived in Section 5 above on the assumption of invariant length intervals. No such hypothesis was made to derive (6.44) and (6.45). Since all rotational motion is at right angles to a line segment specifying the radius of the Earth, the latter quantity is the same in the frames S′ and S even in conventional special relativity where longitudinal length intervals undergo 'length contraction'. This latter circumstance is at the origin of the Ehrenfest paradox [16] of conventional SRT concerning the observations of the diameter and circumference of a rotating disc. This paradox is resolved by Eq. (6.47). Contrary to the assertions of Einstein [40, 41], no modification of Euclidean spatial geometry is required[11].

The HK experiment further demonstrates that observed velocities in different frames are not given correctly by the RRVTR formulae (6.44) and (6.45). In the experiment the parametric velocities are given by the relations (c.f. Eq. (5.14)):

$$\hat{\beta}_{\mathrm{A}}(\mathrm{W-E}) = \frac{\beta_{\mathrm{E}} + \beta'_{\mathrm{A}}}{1 + \beta_{\mathrm{E}}\beta'_{\mathrm{A}}}, \quad (6.48)$$

$$\hat{\beta}_{\mathrm{A}}(\mathrm{E-W}) = \frac{\beta_{\mathrm{E}} - \beta'_{\mathrm{A}}}{1 - \beta_{\mathrm{E}}\beta'_{\mathrm{A}}}. \quad (6.49)$$

It follows from these equations, and (6.40),(6.41) and (6.42) that:

$$\hat{T}_{\mathrm{A}}(\mathrm{W-E}) \equiv \frac{d(\mathrm{W-E})}{\hat{v}_{\mathrm{A}}(\mathrm{W-E})}, = T'(\mathrm{C}')\gamma(\mathrm{C}')^2(1 + \beta_{\mathrm{E}}\beta'_{\mathrm{A}}) \quad (6.50)$$

$$\hat{T}_{\mathrm{A}}(\mathrm{E-W}) \equiv \frac{d(\mathrm{E-W})}{\hat{v}_{\mathrm{A}}(\mathrm{E-W})} = T'(\mathrm{C}')\gamma(\mathrm{C}')^2(1 - \beta_{\mathrm{E}}\beta'_{\mathrm{A}}) \quad (6.51)$$

to be contrasted with the TD relation (6.43) that gives:

$$T_{\mathrm{A}}(\mathrm{W-E}) = T_{\mathrm{A}}(\mathrm{E-W}) = \gamma(\mathrm{C}')T'(\mathrm{C}'). \quad (6.52)$$

It is clear that $\hat{T}_{\mathrm{A}}(\mathrm{W-E})$ and $\hat{T}_{\mathrm{A}}(\mathrm{E-W})$, that show a RS effect between the frames S and S′, since $\hat{T}_{\mathrm{A}}(\mathrm{W-E}) \neq \hat{T}_{\mathrm{A}}(\mathrm{E-W})$, do not respect the TD relations (6.52). Consideration of an experiment where aircraft set out simultaneously for the E−W and W−E equatorial circumnavigations at the same speed and arrive back simultaneously over the ground station containing the Earth-bound clock, shows immediately the fallacious nature of the prediction of (6.50) and (6.51) that $\hat{T}_{\mathrm{A}}(\mathrm{W-E}) \neq \hat{T}_{\mathrm{A}}(\mathrm{E-W})$. The simultaneous arrival back at the ground station of the two aircraft constitutes a triple intersection of the world

[11] Denoting the diameter of the disc by D and its circumference by C, Ehrenfest stated that, because of the LC effect, $C/D < \pi$. Einstein assumed that a ruler used to measure C, unlike the disc itself, undergoes LC, so that if C is the measured circumference, $C/D > \pi$.

lines of the aircraft and the ground station that must be seen as such by an observer in *any* reference frame [9, 12, 42]. Assuming the TD relation between the frames S and S″ on the W−E flight to be:

$$\hat{T}_{\rm A}({\rm W}-{\rm E}) = \hat{\gamma}({\rm C}'')T''({\rm W}-{\rm E}) = \gamma({\rm C}')\gamma'({\rm C}'')(1+\beta_{\rm E}\beta'_{\rm A})T''({\rm W}-{\rm E}) \tag{6.53}$$

and using (6.50) gives:

$$\frac{T''({\rm W}-{\rm E})}{T'({\rm C}')} = \frac{\gamma({\rm C}')}{\gamma'({\rm C}'')} \tag{6.54}$$

so that

$$\Delta T'({\rm W}-{\rm E}) \equiv T'({\rm C}')\left[\frac{T''({\rm W}-{\rm E})}{T'({\rm C}')} - 1\right] = T'({\rm C}')\left[\frac{\gamma({\rm C}')}{\gamma'({\rm C}'')} - 1\right] = \Delta T'({\rm E}-{\rm W}). \tag{6.55}$$

The last member of (6.55) is given by combining $\hat{T}_{\rm A}({\rm E}-{\rm W})$ given by the formula analogous to (6.53) with (6.51). It is then predicted that the values of $\Delta T'$ are the same for the W−E and E−W flights, which is excluded by the results of the HK experiment.

Consideration of the time interval measurements performed in the HK experiment then gives the conclusions:

(i) The concepts of base and travelling frames or the equivalent one of ‘coordinate time’ are essential for a correct analysis of the experiment. The TD effect in a space-time experiment involving three inertial frames, as pointed out by Hafele [39, 37], does not depend solely on the relative velocity of the two observed clocks.

(ii) Observed relative velocities transform according to the RRVTR formulae (6.44) and (6.45), not according to the TDVTR formulae (6.48) and (6.49).

(iii) Lengths parallel to the direction of motion observed in the frames S and S′ are equal (there is no LC). This resolves the Ehrenfest rotating disc paradox [16, 40, 41].

7 Relativistic energy and momentum, $c = V$, and the meaning of Einstein’s second postulate

The relativistic energy, E, and momentum, p, in the frame S, of a ponderable physical object of Newtonian mass, m, at rest in the frame S″ and moving with speed u along the positive x-axis in S, are defined as:

$$E \equiv mV\frac{dx_0}{d\tau''} = m\gamma(\beta_u)V^2 = mU_0V^2, \tag{7.1}$$

$$p \equiv m\frac{dx}{d\tau''} = m\gamma(\beta_u)u = m\beta_u\gamma(\beta_u)V = mU_xV, \tag{7.2}$$

where U is the dimensionless four-vector velocity: $U \equiv (U_0; U_x, 0, 0) \equiv (\gamma(\beta_u); \beta_u\gamma(\beta_u), 0, 0)$. Due to the frame-independent identity (3.25) respected by U, the quantities E, p, m and V are connected by the relation:

$$E^2 = m^2V^4 + p^2V^2. \tag{7.3}$$

This expresses the Lorentz-invariant character of the Newtonian mass in virtue of the identity: $\gamma(\beta_u)^2 - \beta_u^2\gamma(\beta_u)^2 = 1$:

$$E^2 - p^2V^2 = m^2V^4(U_0^2 - U_x^2) = m^2V^4(\gamma(\beta_u)^2 - \beta_u^2\gamma(\beta_u)^2) = m^2V^4. \tag{7.4}$$

Taking the ratio of (7.2) to (7.1) expresses the scaled frame velocity β_u in terms of p, E, and V:

$$\beta_u = \frac{pV}{E}. \tag{7.5}$$

Note that E and p are completely defined, in terms of m, V and the TD factor $\gamma(\beta_u)$ of the frame S'', since (7.2) may be written in an equivalent manner as:

$$p \equiv mV\sqrt{\gamma(\beta_u)^2 - 1}. \tag{7.6}$$

In view of the Lorentz invariance of the Newtonian mass, and the universal character of V, the transformation formulae for relativistic energy and momentum follow directly from the definitions (7.1) and (7.2) and the transformation equations, derived in Section 3 above, for the components of the four-vector velocity U :

$$\gamma(\beta_{u'}) = \gamma(\beta_v)[\gamma(\beta_u) - \beta_v\beta_u\gamma(\beta_u)], \tag{7.7}$$

$$\beta_{u'}\gamma(\beta_{u'}) = \gamma(\beta_v)[\beta_u\gamma(\beta_u) - \beta_v\gamma(\beta_u)] \tag{7.8}$$

obtained by transposing Eqs. (3.20) and (3.23), respectively[12]. They are:

$$\hat{E}' = \gamma(\beta_v)[E - vp] = \gamma(\beta_v)[E - \beta_v pV], \tag{7.9}$$

$$\hat{p}' = \gamma(\beta_v)\left[p - \frac{vE}{V^2}\right] = \gamma(\beta_v)\left[p - \frac{\beta_v E}{V}\right]. \tag{7.10}$$

These transformation equations may be written in a space-time symmetrical [14] manner by introducing an *energy-momentum four vector* with dimensions of energy: $P = (P_0; P_x, 0, 0) \equiv (E; pV, 0, 0)$:

$$\hat{P}'_0 = \gamma(\beta_v)[P_0 - \beta_v P_x], \tag{7.11}$$

$$\hat{P}'_x = \gamma(\beta_v)[P_x - \beta_v P_0]. \tag{7.12}$$

For $\beta_u \ll 1$,

$$\gamma(\beta_u) = \frac{1}{\sqrt{1-\beta_u^2}} = 1 + \frac{\beta_u^2}{2} + \frac{3\beta_u^4}{8} + ... \tag{7.13}$$

so that

$$\begin{aligned} E &= mV^2 + \frac{m\beta_u^2}{2}V^2 + \frac{3m\beta_u^4}{8}V^2 + ... \\ &= mV^2 + \frac{mu^2}{2} + \frac{3m\beta_u^2}{8}u^2 + ... \ . \end{aligned} \tag{7.14}$$

$$\begin{aligned} p &= m\beta_u V + \frac{m\beta_u^3}{2}V + ... \\ &= mu + \frac{m\beta_u^2}{2}u + ... \ . \end{aligned} \tag{7.15}$$

[12] Or, equivalently, by making the subsitutions: $u' \to u$, $u \to u'$ and $v \to -v$ in these equations.

In the Galilean limit $V \to \infty$, $\beta_u \to 0$, the relativistic energy reduces to the kinetic energy of classical mechanics plus a constant 'rest energy' term mV^2 that cancels in all energy-balance equations of classical mechanics on the assumption that the Newtonian mass of each interacting object is conserved. In the same limit, the relativistic momentum is the conventional Newtonian quantity 'mass times velocity'.

To analyse how the classical conservation laws of mass, kinetic energy and momemtum are modified in relativistic kinematics, it will be convenient to consider an inelastic collision between two particles of mass m_1 and m_2 to produce two other particles of mass m_3 and m_4, where, for simplicity, the motion of all four particles is restricted to the same straight line. It is assumed that, intially, the particles 1 and 2 move parallel to the x-axis with speeds u_1, u_2 ($u_1 > u_2$) so that they collide. The speeds of the produced particles along the x-axis are u_3 and u_4 ($u_4 > u_3$). Denoting the energies and momenta of the particles by E_i, p_i ($i = 1, 2, 3, 4$) the total initial (I) and final (F) energies and momenta of the colliding and produced particles are:

$$E_{\rm I} = E_1 + E_2, \quad p_{\rm I} = p_1 + p_2,$$

$$E_{\rm F} = E_3 + E_4, \quad p_{\rm F} = p_3 + p_4.$$

Since no external forces act on the particles, Newton's First Law implies that $p_{\rm I} = p_{\rm F}$. For this it is necessary that 'force' in Newton's Second Law is defined as the time derivative of the relativistic momentum, and that the 'action' in Newton's Third Law is identified with a force equal to the time derivative of the relativistic momentum of the 'acting' particle. It then follows, from this generalisation of Newton's mechanical laws, that $p_{\rm I} = p_{\rm F}$ in any inertial frame. It will now be found instructive to transform the energies and momenta of the particles into a frame, S$'$, in which the total momentum of the incoming particles is zero: $\vec{\hat{p}}'_1 = -\vec{\hat{p}}'_2$. From (7.10), the corresponding value of β_v is given by the equation:

$$\vec{\hat{p}}'_{\rm I} = \gamma(\beta_v)\left[p_{\rm I} - \frac{\beta_v E_{\rm I}}{V}\right] = 0 \tag{7.16}$$

so that

$$\beta_v = \frac{V p_{\rm I}}{E_{\rm I}} = \frac{V(p_1 + p_2)}{E_1 + E_2} = \frac{m_1\gamma_1\beta_1 + m_2\gamma_2\beta_2}{m_1\gamma_1 + m_2\gamma_2}, \tag{7.17}$$

where $\beta_{1,2} = u_{1,2}/V$ and $\gamma_{1,2} = 1/\sqrt{1 - \beta_{1,2}^2}$. In the frame S$'$ which is, by definition, the centre-of-mass (CM) frame of the collision of particles 1 and 2, the total initial energy, $\hat{E}'_{\rm I}$ is given by:

$$\hat{E}'_{\rm I} \equiv W_{\rm I} V^2 = \hat{E}'_1 + \hat{E}'_2. \tag{7.18}$$

Similarly, the CM energy of particles 3 and 4 is:

$$\hat{E}'_{\rm F} \equiv W_{\rm F} V^2 = \hat{E}'_3 + \hat{E}'_4, \tag{7.19}$$

where

$$\hat{E}'_i = \gamma(\beta_v)[E_i - v p_i] \quad i = 1, 2, 3, 4. \tag{7.20}$$

The inverse of Eq. (7.16) and the analogous equation for $p_{\rm F}$ and $E_{\rm F}$ gives:

$$p_{\rm I} = W_{\rm I}\beta_v\gamma(\beta_v)V, \quad p_{\rm F} = W_{\rm F}\beta_v\gamma(\beta_v)V \tag{7.21}$$

so that momentum conservation in the frame S: $p_{\rm F} = p_{\rm I}$ requires conservation of CM energy in the collision process; $W_{\rm I}V^2 = W_{\rm F}V^2 \equiv WV^2$. The inverse of the energy transformation equation then gives:

$$E_{\rm I} = \gamma(\beta_v)W_{\rm I}V^2 = \gamma(\beta_v)WV^2 = \gamma(\beta_v)W_{\rm F}V^2 = E_{\rm F} \qquad (7.22)$$

so that the relativistic energy is also conserved in the frame S. Combining (7.21), (7.22), the equality $W_{\rm I} = W_{\rm F} = W$ and the identity $\gamma(\beta_v)^2 - \beta_v^2\gamma(\beta_v)^2 \equiv 1$ yields the relation:

$$\begin{aligned} W^2V^4 &= (\hat{E}'_1 + \hat{E}'_2)^2 = (\hat{E}'_3 + \hat{E}'_4)^2 \\ &= (E_1 + E_2)^2 - (p_1 + p_2)^2 = (E_3 + E_4)^2 - (p_3 + p_4)^2 \end{aligned} \qquad (7.23)$$

showing that the 'effective mass', W, of the collision process is an invariant quantity, analogous to the Newtonian mass, m, of a single physical object in (7.4), given, in all inertial frames, by the expressions in the last line of (7.23). It is now seen that the conservation of mass, kinetic energy and momentum of Newtonian kinematics are replaced, in relativistic kinematics, by conservation of relativistic energy and momentum and invariance of the effective mass W that characterises the scattering process.

The CM energies of particles produced in a generic inelastic process:

$$1 + 2 \rightarrow 3 + 4 + 5 + ... + \mathrm{N}$$

give, by generalisation of (7.19):

$$WV^2 = \hat{E}'_3 + \hat{E}'_4 + \hat{E}'_5 + ... + \hat{E}'_{\rm N} \qquad (7.24)$$

from which may be derived, in a manner analogous to (7.23) the relation

$$W^2V^4 = (E_1 + E_2)^2 - (p_1 + p_2)^2 = (\sum_{i=3}^{\rm N} E_i)^2 - (\sum_{i=3}^{\rm N} \vec{p}_i)^2 \qquad (7.25)$$

valid for arbitary N and arbitary directions for the momenta of the final state particles.

With a view to the subsequent discussion of photon kinematics, the collinear collision process:

$$1 + 2 \rightarrow 3 + 4$$

$(u_1 > u_2,\ u_4 > u_3)$, for the case of equal particle masses:

$$m_1 = m_2 = m_3 = m_4 \equiv m$$

will now be considered. The velocity of the CM system is then given by Eq. (7.17) as:

$$\beta_v = \frac{\gamma_1\beta_1 + \gamma_2\beta_2}{\gamma_1 + \gamma_2}. \qquad (7.26)$$

The TDVTR (5.13) gives, together with (7.26) the parametric velocities of the particles in the CM system:

$$\hat{\beta}'_1 = \hat{\beta}'_4 = -\hat{\beta}'_2 = -\hat{\beta}'_3 \equiv \beta' = \frac{\gamma_1\gamma_2(\beta_1 - \beta_2)}{1 + \gamma_1\gamma_2(1 - \beta_1\beta_2)}. \qquad (7.27)$$

The equality of the masses and the magnitudes of the particle velocities in the CM system implies all transformed energies and moduli of momenta in the CM system, S$'$, are equal:

$$\hat{E}'_1 = \hat{E}'_2 = \hat{E}'_3 = \hat{E}'_4 = m\gamma(\beta')V^2 = \frac{WV^2}{2}, \tag{7.28}$$
$$\hat{p}'_1 = \hat{p}'_4 = -\hat{p}'_2 = -\hat{p}'_3 = m\beta'\gamma(\beta')V. \tag{7.29}$$

On transforming the energies and momenta of the scattered particles back into the base frame S it is seen that

$$E_4 = E_1, \quad p_4 = p_1, \quad \beta_4 = \beta_1, \tag{7.30}$$
$$E_3 = E_2, \quad p_3 = p_2, \quad \beta_3 = \beta_2. \tag{7.31}$$

Four different space-time experiments involving the same equal mass collinear collision process, just described, will now be considered:

I Particles 1 and 2 have frame velocities β_1 and β_2 in the base frame S.

II Particles 1 and 2 have frame velocities $\beta'_1 = \beta'$, $\beta'_2 = -\beta'$ in the travelling frame S$'$.

III Particles 1 and 2 have the frame velocities $\beta_1 = -\beta'$, $\beta_2 = \beta'$ in the base frame S.

IV Particles 1 and 2 have the frame velocities $\beta'_1 = -\beta_1$, $\beta'_2 = -\beta_2$ in the travelling frame S$'$

The velocities of particles 1 and 2 in experiments I, II or III, IV are related by the TDVTR of Eq. (5.13). The observed velocities of the scattered particles, as predicted by the RRVTR, in these four distinct experiments are presented in Table 1.

As well as the relevant formulae, numerical values are given for fixed input parameters $\beta_2 = 0.5$ $\beta_v = \sqrt{3}/2$ (corresponding to $\gamma(\beta_v) \equiv \gamma_v = 2$, $\beta_1 = 0.97$ and $\beta' = 0.646$. Also shown in brackets are the values $\beta_2 = 0.5$, $\beta_v = 0.735$, $\beta_1 = 0.97$ and $\beta' = 0.235$ appropriate to Galilean kinematics, where both the RRVTR and TDVTR are replaced by the same classical velocity addition formula. For example, in I: $\tilde{\beta}'_3 = \beta_2 - \beta_v$, $\tilde{\beta}'_4 = \beta_1 - \beta_v$. Marked differences are seen between the observed velocities presented in the second column of Table 1 and the parametric velocities shown in the third column. Some observed velocities are superluminal both in the base frame (particle 4 in experiment II and particle 3 in experiment IV) and in the travelling frame (particle 4 in experiment III). Also the sign of the observed velocity may be different both from the corresponding parametric velocity and that given by the Galilean calculation (particle 4 in experiment IV).

It is important to now specify the precise the operational meaning of the observed and parametric velocities shown in Table 1. In all cases the frame velocities of the incident particles are fixed initial parameters and the kinematical calculation of the scattered velocities is performed in a particular frame: S for I and III, S$'$ for II and IV. The observed velocities of the scattered particles are then simply those that would be measured by particle detectors situated in the frame in which the velocities of the incident particles are specified, and the kinematical calculation of the scattered velocities is performed.

Frame velocity of 1 and 2	Observed velocity of 3 and 4	Parametric velocity of 3 and 4
I β_1 β_2 0.97 0.5 (0.97) (0.5)	$\tilde{\beta}_3 = \beta_2$, $\tilde{\beta}_4 = \beta_1$ 0.5 0.97 $\tilde{\beta}'_3 = -\gamma_v(\beta_v - \beta_2)$, $\tilde{\beta}'_4 = \gamma_v(\beta_1 - \beta_v)$ -0.732 0.208 (-0.235) (0.235)	$\hat{\beta}'_3 = -\frac{(\beta_v-\beta_2)}{1-\beta_v\beta_2}$, $\hat{\beta}'_4 = \frac{(\beta_1-\beta_v)}{1-\beta_v\beta_1}$ -0.646 0.646
II $\beta'_1 = \beta'$, $\beta'_2 = -\beta'$ 0.646 -0.646 (0.235) (-0.235)	$\tilde{\beta}_3 = -\frac{\beta'}{\gamma_v} + \beta_v$, $\tilde{\beta}_4 = \frac{\beta'}{\gamma_v} + \beta_v$ 0.543 1.189 (0.5) (0.97) $\tilde{\beta}'_3 = -\beta'$, $\tilde{\beta}'_4 = \beta'$ -0.646 0.646	$\hat{\beta}_3 = \frac{(\beta_v-\beta')}{1-\beta_v\beta'}$, $\hat{\beta}_4 = \frac{(\beta_v+\beta')}{1+\beta_v\beta'}$ 0.5 0.97
III $\beta_1 = -\beta'$, $\beta_2 = \beta'$ -0.646 0.646 (-0.235) (0.235)	$\tilde{\beta}_3 = \beta'$, $\tilde{\beta}_4 = -\beta'$ 0.646 -0.646 $\tilde{\beta}'_3 = -\gamma_v(\beta_v - \beta')$, $\tilde{\beta}'_4 = -\gamma_v(\beta_v + \beta')$ -0.441 -3.023 (-0.5) (-0.97)	$\hat{\beta}'_3 = -\frac{(\beta_v-\beta')}{1-\beta_v\beta'}$, $\hat{\beta}'_4 = -\frac{(\beta_v+\beta')}{1+\beta_v\beta'}$ -0.5 -0.97
IV $\beta'_1 = -\beta_1$, $\beta'_2 = -\beta_2$ -0.97 -0.5 (-0.97) (-0.5)	$\tilde{\beta}_3 = -\frac{\beta_2}{\gamma_v} + \beta_v$, $\tilde{\beta}_4 = -\frac{\beta_1}{\gamma_v} + \beta_v$ 1.116 0.381 (0.235) (-0.235) $\tilde{\beta}'_3 = -\beta_2$, $\tilde{\beta}'_4 = -\beta_1$ -0.5 -0.97	$\hat{\beta}_3 = \frac{(\beta_v-\beta_2)}{1-\beta_v\beta_2}$, $\hat{\beta}_4 = -\frac{(\beta_1-\beta_v)}{1-\beta_v\beta_1}$ 0.646 -0.646

Table 1: *Observed and parametric velocities in space-time experiments I, II, III and IV specified by the frame velocities presented in the first column, for the collinear equal-mass scattering process:* $1+2 \to 3+4$. *Numbers in round brackets refer to Galilean kinematics.* $\beta_v = \sqrt{3}/2$, $\gamma_v = 2$, $\beta' = 0.646$. *For Galilean kinematics* $\beta_v = 0.735$, $\beta' = 0.235$. *See text for discussion.*

This is the way that all particle velocities have hitherto been calculated or measured in experiments performed in laboratories at rest on the surface of the Earth. The situation is then analogous to that in experiments I and III where the input velocities are specified in the base frame S. In order to observe the corresponding travelling frame velocities, it is necessary to equip the particle detectors in S with some device, such as a flash-lamp, which produces a signal which can be observed from the travelling frame at the epochs at which the particles are detected in S. The observed velocity in the travelling frame S′ is then defined as the spatial sparation beween the signal events divided by the time interval between these events in the same frame, corrected, if necessary, for signal propagation delays.

In the case that the initial velocities and kinematics are specified in the travelling frame, as in experiments II and IV, the travelling frame particle detectors must be similarly equipped with signalling devices visible from the base frame S, and the observed velocites are similarly defined in terms of space and time intervals between signal events observed in S corresponding to the passage of a particle between detectors in the travelling frame. No such kinematical experiment, performed by particle detectors in a frame moving a speeds $\simeq V$ relative to an external observer, or by an external observer moving at such speeds relative to the surface of the Earth, has hitherto been performed, so there has been no possiblity, to date, to check the predicted and, possibly superluminal, velocities $\tilde{\beta}$ or $\tilde{\beta}'$ of Table 1, as in the experiments II, III and IV.

Inspection of the entries in the last column of Table 1 shows that the parametric velocities obtained by transforming the frame velocities of the particles 3 and 4 according to the TDVTR, do not give observed velocities in the same space-time experiment, (these are shown in the second column) but instead kinematical configurations of different space-time experiments. As is also the case for particles 1 and 2, the experiments that are related in this way are I and II or III and IV.

Einstein's second postulate of special relativity, as well as its complete physical interpretation, follows on consideration of experiments similar to II and III above, where the scattered particles 3 and 4 are replaced by photons. For example, they could be produced by the decay of para-positronium via the annhilation process: $e^+e^- \rightarrow \gamma\gamma$. In the experiment similar to II, the decaying positronium atom is at rest in the travelling frame S′, and the decay photons propagate along the positive and negative x'-axes, whereas in the experiment similar to III the positronium is at rest in the base frame S, while the decay photons propagate along the positive and negative x-axes. The photon velocities in these experiments are then the same as for the case of collinear scattering of massless particles in the experiments II and III just discussed. The corresponding photon velocities, presented in Table 2, are obtained from those in Table 1 by taking the limit $m_3, m_4 \rightarrow 0$ so that, according to Eq. (7.3), $E_{3,4} \rightarrow p_{3,4}V^2$. Then from (7.5), $\beta_{3,4} \rightarrow 1$ or the speed of light in free space $\equiv c \rightarrow V$. It is convenient to assume that the photon is strictly massless: $m_3 = m_4 = 0$, so that $c = V$. However as will be discussed below, such a postulate leads to conceptual problems when the kinematics of massless photons of different energy in the same frame is considered. In the case of observations of positronium decay in different frames, presented in Table 2, it is sufficient to assume that any non-vanishing mass, μ, of the photon is much less than the mass of an electron.

Observed velocities of 3 and 4	Parametric velocities of 3 and 4
Experiment II $\tilde{\beta}_3 = -\frac{1}{\gamma_v} + \beta_v, \quad \tilde{\beta}_4 = \frac{1}{\gamma_v} + \beta_v$ 0.366 1.366 $\tilde{\beta}'_3 = -1, \quad \tilde{\beta}'_4 = 1$	$\hat{\beta}_3 = -1, \quad \hat{\beta}_4 = 1$
Experiment III $\tilde{\beta}_3 = 1, \quad \tilde{\beta}_4 = -1$ $\tilde{\beta}'_3 = -\gamma_v(\beta_v + 1), \quad \tilde{\beta}'_4 = -\gamma_v(\beta_v - 1)$ -3.732 0.268	$\hat{\beta}'_3 = 1, \quad \hat{\beta}'_4 = -1$

Table 2: *Observed and parametric velocities in space-time experiments II, III of Table 1 for for the collinear scattering:* $1 + 2 \rightarrow 3 + 4$ *of massless particles, or positronium decay:* $e^+e^- \rightarrow \gamma\gamma$*. In Experiment II,* $\beta'_1 = 1$*,* $\beta'_2 = -1$*. In Experiment III,* $\beta_1 = -1$*,* $\beta_2 = 1$*.* $\beta_v = \sqrt{3}/2$*,* $\gamma_v = 2$*. See text for discussion.*

Reflection on the entries of Table 2, together with the operational definitions of the oberved and parametric velocities, as described above, reveals to what extent Einstein's postulate:

`'The speed of light is the same in all inertial frames and independent of that of its source.'`

is consistent with predictions given by the kinematics of massless photons, and their observation in different inertial frames. If $\mu V^2 \ll E$ so that, from (7.3), $E \simeq pV$ it is clear from (7.5) that the frame velocity, c, of a photon is given by $c \simeq V$ in all inertial frames, and that $c = V$ when $\mu = 0$. These relations hold in an arbitary inertial frame and for arbitary boosts of the CM frame of photon production process. This is because the relations $E = pV$, $c = V$ hold when $\mu = 0$ regardless of the nature and kinematical configuration (boost of CM frame) of the photon production process. It is clear, however, from the entries in the first row of Table 2, that the *observed speed* of a photon produced by positronium decay in the travelling frame S′ but observed from the base frame S, is not equal to V. Also, the *observed speed* of a photon from positronium decay in the base frame S, as viewed from the travelling frame S′ (the fourth row of Table 2) is not equal to V. Thus the feature of conventional special relativity theory most at variance with commonsense experience —a light signal that has the same velocity when viewed from different inertial frames in relative motion— does not occur. Such behaviour is a prediction of the TDVTR, describing the transformation of parametric velocities, and relating configurations in different space-time experiments, not of the RRVTR that describes the transformation of observed relative velocities between different inertial frames in the same space-time experiment. The RRVTR for massless particles shown in Table 2 were first derived in connection with with an analysis of the Michelson-Morley experiment [1, 2]. It is seen that the constancy of the frame velocity of a photon and its independence of source velocity is a necessary consequence of relativitistic kinematics and of the postulate that light consists of massless particles —Einstein's 'light quanta', later called photons. Similar behaviour is manifested by any massless particle. This seems unnatural only if light propagation in free space is associated, not with free particle motion in accordance with Newton's First Law, but with the propagation of wave motion in a material aether, that provides a preferred frame. Although Einstein's discovery of the particulate nature of light (for which he was later awarded the Nobel Prize) was published in the same year in which the original special relativity paper was written and published, Einstein never noticed (in spite of his rejection of the aether hypothesis) the direct connection between his light quantum concept and the second postulate of special relativity [13]. Indeed, further consideration of the actual photonic nature of 'electromagnetic waves' enables many key concepts of quantum mechanics to be derived, from first principles, in a simple and transparent manner [43].

Many different experiments have demonstrated the constancy of the speed of light, over a wide range of photon energies, indicating that the mass, μ, of the photon must be very small. The best upper limit from this type of experiment is given by use of the lowest possible photon energy (the highest wavelength). For example, an experiment with $E_\gamma = 5.7 \times 10^{-5}$ eV ($\lambda_\gamma = 1.73$ m) [44] found that, according to the relation:

$$\beta(E_\gamma, \mu) = \sqrt{1 - \left(\frac{\mu V^2}{E_\gamma}\right)^2} \tag{7.32}$$

given by combining (7.3) and (7.5), the experimental precision of one part in 10^5 on the measurement of the speed, $c = V$, of such photons corresponds to the limit [45]:

$$\mu \le 3 \times 10^{-9} \text{ eV}.$$

This limit implies that experiments with higher photon energies must have smaller values of $\Delta\beta \equiv 1 - \beta$ according to the relation (valid for small $\mu V^2/E_\gamma$):

$$\Delta\beta(E_\gamma, \mu) \equiv 1 - \beta(E_\gamma, \mu) \simeq \frac{1}{2}\left(\frac{\mu V^2}{E_\gamma}\right)^2. \tag{7.33}$$

The relativistic formula for the energy of an object of Newtonian mass m, Eq. (7.1) leads to conceptual problems at the limit $m = 0$. Suppose that two particles of the same Newtonian mass are produced in the same inertial frame by different physical processes, or by the same physical process with a different Lorentz boost. Denoting their energies by $E_{\rm U}$, $E_{\rm L}$ ($E_{\rm U} > E_{\rm L}$) it follows from (7.1) that

$$R \equiv \frac{E_{\rm U}}{E_{\rm L}} = \sqrt{\frac{1-\beta_L^2}{1-\beta_U^2}} > 1. \tag{7.34}$$

Notice that the Newtonian mass, m, of the particles cancels in this energy ratio. The first member of (7.34) can be rearranged to give:

$$\begin{aligned} \beta_U - \beta_L &= \frac{1}{\beta_U + \beta_L}\left[1 - \frac{1}{R^2}\right](1 - \beta_L^2) \\ &\simeq \frac{1}{\beta_U + \beta_L}\left[1 - \frac{1}{R^2}\right]\left(\frac{mV^2}{E_L}\right)^2, \end{aligned} \tag{7.35}$$

where Eq. (7.32) has been used. The inequality in the last member of (7.34) requires that

$$\beta_U > \beta_L. \tag{7.36}$$

Since $m = 0$ in (7.35) necessarily requires that $\beta_U = \beta_L$, which contradicts (7.34) and (7.36), it follows that (7.34)-(7.36) can hold only if

$$m > 0. \tag{7.37}$$

Thus the observation of two identical particles with different energies in the same inertial frame, together with the definition in Eq. (7.1) of relativistic energy, is incompatible with a massless nature for the particle.

Applied to photons, the energy of which may vary over many orders of magnitudes without any observable change in their frame velocity, this argument implies that the strict application of the postulate of the constancy of the speed of light, must break down, even as applied to frame or parametric velocities, if the formulae of relativistic kinematics hold good. In fact, the definition of relativistic energy in Eq. (7.1) allows all finite positive values of E for any fixed finite and non-vanishing value of m *no matter how small* since β_u and $\gamma(\beta_u)$ are physical in the ranges:

$$0 < \beta_u < 1, \qquad 1 < \gamma(\beta_u) < \infty.$$

If $\beta_u = 1$, then $\gamma(\beta_u) = \infty$ but the energy is indeterminate, and the possibility of the existence of a value of β_u less than unity, corresponding to any finite positive value of E, is lost.

Given the stringent upper limits on the mass of the photon that have been experimentally established, the breakdown of Einstein's second postulate required by the self-consistency of relativistic kinematics has no practical consequences. Assuming a photon mass equal to the currently quoted upper limit [46] $\mu V^2 = 10^{-18}$ eV, the range of β needed to accomodate the observation of photons in the energy range:

$$5.7 \times 10^{-5}\ \text{eV} < E_\gamma < 1\ \text{TeV}$$

is given by (7.35):

$$\begin{aligned} \beta_U - \beta_L &\simeq \frac{1}{2}\left[1 - \frac{1}{R^2}\right]\left(\frac{\mu V^2}{E_L}\right)^2 \\ &= \frac{1}{2}\left[1 - 3.26 \times 10^{-33}\right]\left(\frac{10^{-18}}{5.7 \times 10^{-5}}\right)^2 \\ &= 1.54 \times 10^{-28} \end{aligned} \tag{7.38}$$

to be compared with the precision of 3.3×10^{-9} implied by the least significant digit of the *defined value* [46] of the speed of light in free space:

$$c = 299\ 792\ 458\ \text{m/s}.$$

It is shown in Eq. (7.38) that with a photon mass of 10^{-18} eV, sixteen orders of magnitude in the range of E_γ (and therefore of $\gamma(\beta_u)$) are accomodated by a variation of only 1.54×10^{-28} in $\Delta\beta_u$. In fact

$$1 \times 10^{-60} < 1 - \beta_u < 1.54 \times 10^{-28},$$

so that $1-\beta_u$ varies over 31 orders of magnitude for the observed 16 orders of magnitude variation of E_γ. That the frame velocity of light may be taken as constant, equal to the limiting velocity V, is therefore an excellent approximation, given the existing experimental upper limit on the photon mass.

The important conceptual difference between the operational meanings of the RRVTR, that relates velocities observed in different reference frames in the same space-time experiment and the TDVTR that gives transformations between kinematical configurations of different space-time experiments, that was evident in the analysis of the HK experiment in the previous section, may be further illustrated by consideration of observations of unstable elementary particles in different inertial frames [7].

Consider, for example, a charged pion that is produced in a target with an energy of 10 GeV that subsequently travels a distance of 100 m in a secondary beam line before colliding with a proton in a liquid hydrogen target. The laboratory frame velocity of the pion is $\vec{v}$ where $v \equiv |\vec{v}| = 0.999990255c$ corresponding to $\gamma(\beta_v) = 71.83$. Application of the TDVTR (3.22) between the laboratory frame and the rest frame of the pion gives the kinematical configuration of the reciprocal experiment where pion is at rest and the proton, with which it collides, has velocity $-\vec{v}$. The physical independence of the primary and reciprocal experiment is made manifest by calculating the probabilities that the pion will decay before interacting with the proton in the two experiments. The mean rest-frame lifetime of the pion is $\tau_\pi = 2.6 \times 10^{-8}$ s [46]. Due to time dilation, the laboratory-frame mean lifetime is $\tau_{\rm LAB} = \gamma(\beta_v)\tau_\pi$. The probability therefore that the pion will survive to interact with the proton after its 100 m flight path, L_π, is $\exp[-L_\pi/\{v(\gamma(\beta_v)\tau_\pi)\}] = 0.84$. In the reciprocal experiment, the time of flight of the proton is equal to that of the the pion in the primary experiment, but since the pion is at rest there is no TD effect. The survival probabilty of the pion is then $\exp[-L_\pi/v\tau_\pi] = 2.7 \times 10^{-6}$. In the primary experiment the base frame is that of the laboratory, the travelling frame the proper frame of the pion. In the latter frame the speed of the proton is given by the RRVTR (5.4) as $v\gamma(\beta_v)$ giving the the survival probability: $\exp[-L_\pi/\{(v\gamma(\beta_v))\tau_\pi\}] = 0.84$, the same as in the base frame. In the reciprocal experiment, the speed of the pion in the proton rest frame (the travelling frame) is also $v\gamma(\beta_v)$, however the TD effect is now inverted: $\tau' = \tau_\pi/\gamma(\beta_v)$ giving the survival probability in the travelling frame: $\exp[-L_\pi/\{(v\gamma(\beta_v))\tau'\}] = \exp[-L_\pi/\{v\tau_\pi\}] = 2.7 \times 10^{-6}$, the same as calculated in the pion rest frame (the base frame). This example gives another graphic illustration that it is the RRVTR, not the TDVTR, that gives correctly the transformation of relative velocities in the same space-time experiment. As is clear from inspection of the entries in the third column of Table 1, the TDVTR instead relates kinematical configurations of *physically independent* space-time experiments: I is related to II and III to IV. The latter behaviour is shared by the Lorentz transformations of energy and momentum.

8 Summary, discussion and outlook

The space-time physics of an experiment in which a clock C, at rest in the base frame S, is compared with a clock C′ at rest in a travelling frame S′, is described by the equations (c.f. (6.1), (6.2)):

$$\begin{aligned} x'(\mathrm{C}') &= D', && (8.1)\\ x(\mathrm{C}') &= D'+vt(\mathrm{C}), && (8.2)\\ t(\mathrm{C}) &= \gamma(\beta_v)\tau(\mathrm{C}'). && (8.3) \end{aligned}$$

Because of the freedom of choice of spatial coordinate systems in S and S', which does not affect any physical prediction, it is always possible to choose the origin of spatial coordinates in S such that (8.2) holds. The corresponding equations for two spatially-separated clocks C'_1, C'_2 are:

$$\begin{aligned} x'(\mathrm{C}'_1) &= D'_1, \quad x'(\mathrm{C}'_2) = D'_2, && (8.4)\\ x(\mathrm{C}'_1) &= D'_1+vt(\mathrm{C}), \quad x(\mathrm{C}'_2) = D'_2+vt(\mathrm{C}), && (8.5)\\ t(\mathrm{C}) &= \gamma(\beta_v)\tau(\mathrm{C}'_1) = \gamma(\beta_v)\tau(\mathrm{C}'_2). && (8.6) \end{aligned}$$

The TD relations in (8.6) show that C'_1 and C'_2 are synchronised in both S and S$'$, while (8.4) and (8.5) give:

$$L' \equiv x'(\mathrm{C}'_2) - x'(\mathrm{C}'_1) = D'_2 - D'_1 = x(\mathrm{C}'_2) - x(\mathrm{C}'_1) \equiv L. \qquad (8.7)$$

There are therefore no RS or LC effects. How these arise in conventional special relativity, from misinterpretation of space-time coordinate symbols in the LT is explained in Section 4 above and in Refs. [1, 2, 26, 27, 28].

The physically independent reciprocal experiment, in which the clock C$'$ is a rest in the base frame S$'$, while C is at rest in the travelling frame S that moves with speed v aong the negative x' axis, is described by equations similar to (8.1)-(8.3) with the replacements: $x'(\mathrm{C}') \to x(\mathrm{C})$, $x(\mathrm{C}') \to x'(\mathrm{C})$, $t(\mathrm{C}) \to t'(\mathrm{C}')$, $\tau(\mathrm{C}') \to \tau(\mathrm{C})$, $D' \to D$ and $v \to -v$. The equations for the primary experiment contain only the spatial coordinates of the clock C$'$, those of the reciprocal experiment only the spatial coordinates of the clock C.

A particular space-time experiment involving three clocks C, C$'$ and C$''$ at rest in the frames S, S$'$ and S$''$ respectively, is one in which the initial conditions are such that C$''$ has the frame velocity u' along the positive x' axis in S$'$ and C$'$ frame velocity v along the positive x axis in the base frame S. The TD relation for the clock C$''$, as viewed from S, is (c.f. Eqs. (6.13), (6.29)):

$$\begin{aligned} t(C) &= \hat{\gamma}(\mathrm{C}'')\tau(\mathrm{C}''), && (8.8)\\ \hat{\gamma}(\mathrm{C}'') &= \gamma(\mathrm{C}')\{\gamma'(\mathrm{C}'') + \beta(\mathrm{C}')[\beta'(\mathrm{C}'')\gamma'(\mathrm{C}'')]\}. && (8.9) \end{aligned}$$

Eq. (8.9) gives the transformation of the TD factor for the clock C$''$ between the frames S$'$ and S. In the experiment described by (8.8) and (8.9), the TD effect for the clock C$''$ relative to C$'$ in the travelling frame S$'$ is not determined by the velocity of C$''$ relative to C$'$ as in the simple two-clock experiment with the TD relation (8.3). It is instead given by the relation (c.f. Eqs. (6.19),(6.30)):

$$t'(\mathrm{C}') = \bar{\gamma}'(\mathrm{C}'')\tau(\mathrm{C}'') \equiv \frac{\hat{\gamma}(\mathrm{C}'')}{\gamma(\mathrm{C}')}\tau(\mathrm{C}''), \qquad (8.10)$$

where (8.9) gives (c.f. Eq. (6.31))

$$\bar{\gamma}'(\mathrm{C}'') = \gamma'(\mathrm{C}'')[1+\beta(\mathrm{C}')\beta'(\mathrm{C}'')]. \qquad (8.11)$$

The predictions of Eqs. (8.10) and (8.11) as well as the importance of the concepts of base and travelling frames have been experimentally confirmed by the HK experiment [15]. In this experiment the hypothetical clock C records an unobserved 'coordinate time' in the ECI frame (base frame) comoving with the centroid of the Earth, whereas C′ and C″ respectively are, respectively, the Earth-bound and airborne clocks in the travelling frames S′ and S″. The initial velocity parameter v is determined by the speed of rotation of the Earth, u' by the speed of the aircraft relative to the surface of the Earth.

The fundamental space-time equations in special relativity are, firstly, TD relations such as (8.3), (8.6) or (8.8), and, secondly, the transformation equation (8.9) for the TD factor γ. Only times and velocities, not spatial coordinates, appear in these equations. The TD effect is thus a universal (position-independent) transformation of time intervals as observed in two inertial frames. The equations containing spatial coordinates, (8.1) and (8.2), which are the equations of motion (or world lines) of the clock C′ in the frames S′ and S, repectively, do not depend on the limiting velocity V, and are the same as in Galilean relativity. The transformation equation (8.9) of the TD factor may also be written as the LT of the temporal component of the dimensionless four-vector velocity of the clock C″:

$$U_0 = \gamma(\mathrm{C}')[U'_0 + \beta(\mathrm{C}')U'_x], \tag{8.12}$$

where

$$U' = (U'_0; U'_x, 0, 0) \equiv (\gamma'(\mathrm{C}''); \beta'(\mathrm{C}'')\gamma'(\mathrm{C}''), 0, 0).$$

As shown in Eqs. (3.22) and (3.23), (8.12) is algebraically equivalent to the LT of the spatial component of U':

$$U_x = \gamma(\mathrm{C}')[U'_x + \beta(\mathrm{C}')U'_0] \tag{8.13}$$

or the TDVTR:

$$\hat{\beta}(\mathrm{C}'') = \frac{\beta(\mathrm{C}') + \beta'(\mathrm{C}'')}{1 + \beta(\mathrm{C}')\beta'(\mathrm{C}'')}. \tag{8.14}$$

The transformation equations of relativistic energy: $E = mU_0V^2$ and momentum: $p = mU_xV$ are direct consequences of (8.12)-(8.14).

The 'parametric velocity', $\hat{\beta}(\mathrm{C}'')$, on the left side of (8.14) which specifies the TD factor $\hat{\gamma}(\mathrm{C}'')$ of Eq. (8.9) according to the relation:

$$\hat{\gamma}(\mathrm{C}'') = \gamma(\hat{\beta}(\mathrm{C}'')) \tag{8.15}$$

should not be confused with the observed velocity $\tilde{\beta}(\mathrm{C}'')$ of C″ in S as given by the RRVTR Eq. (5.12) (see Figs. 4-7 and Table 1):

$$\tilde{\beta}(\mathrm{C}'') = \frac{\beta'(\mathrm{C}'')}{\gamma(\mathrm{C}')} + \beta(\mathrm{C}'). \tag{8.16}$$

The erroneous conclusions obtained hitherto, concerning space-time physics, arising from misinterpretation of the Lorentz transformation equations, largely originate from insufficient attention being paid to the actual operational meaning, in physical measurements or observations, of the mathematical symbols which appear in the equations. Predictions are obtained which do follow in a mathematically rigorous way from the equations, but which are irrelevant to the actual description of physical rulers and clocks. It is instructive, as an illustration of this, to compare the physical predictions (8.3) of the TD effect and (8.9) for the transformation law of the TD factor with some purely mathematical properties, that may be derived from the LT, of an arbitary, generic, four-vector: $G \equiv (G_0; G_x, G_y, G_z)$. The latter is, by definition, a four-vector if it transforms between the frames S and S′ according to the equations:

$$\begin{aligned} G'_x &= \gamma(\beta_v)[G_x - \beta_v G_0], && (8.17) \\ G'_0 &= \gamma(\beta_v)[G_0 - \beta_v G_x] && (8.18) \end{aligned}$$

while $G'_y = G_y$ and $G'_z = G_z$. For comparison, the similar space-time transformation equations for the clock C′, equivalent to (8.1)-(8.3) are[13]:

$$\begin{aligned} \mathrm{x}'(\mathrm{C}') &= \gamma(\beta_v)[\mathrm{x}(\mathrm{C}') - v\mathrm{t}(\mathrm{C})] = 0, && (8.19) \\ \mathrm{t}'(\mathrm{C}') &= \gamma(\beta_v)[\mathrm{t}(\mathrm{C}) - \frac{\beta_v \mathrm{x}(\mathrm{C}')}{V}]. && (8.20) \end{aligned}$$

The transformation equation of the TD factor $\gamma'(\mathrm{C}'')$ is algebraically equivalent to the LT (8.12) or (8.13) which are formally identical to the inverses of the generic transformations (8.18) and (8.17) respectively.

As is well known, the transformation equations (8.17) and (8.18), together with the definition (3.18) of $\gamma(\beta_v)$, lead to the existence of an invariant, Γ, associated with the four-vector G[14]:

$$\Gamma^2 = G_0^2 - G_x^2 = (G'_0)^2 - (G'_x)^2. \quad (8.21)$$

If $\Gamma^2 > 0$, $G_0 > G_x$ and a value of β_v can always be chosen such that $G'_x = 0$, but G'_0 given by (8.18) can never vanish, for any value of β_v. In this case G is termed a 'time-like' four-vector. On the contrary, if $\Gamma^2 < 0$, then $G_x > G_0$ and a value of β_v can always be chosen such that $G'_0 = 0$, but G'_x can never vanish, and G is called 'space-like'. Identifying the components of G with spatial and temporal intervals: $G_{\mathrm{x}} = \Delta\mathrm{x}$, $G_0 = V\Delta\mathrm{t}$ then gives:

$$\text{time} - \text{like interval}$$

$$\Gamma^2 = V^2(\Delta\mathrm{t})^2 - (\Delta\mathrm{x})^2 = V^2(\Delta\mathrm{t}')^2 - (\Delta\mathrm{x}')^2 > 0, \quad (8.22)$$

$$\text{space} - \text{like interval}$$

$$-\Gamma^2 = (\Delta\mathrm{x})^2 - V^2(\Delta\mathrm{t})^2 = (\Delta\mathrm{x}')^2 - V^2(\Delta\mathrm{t}')^2 > 0. \quad (8.23)$$

Examination of (8.23) seems to give the possibility of a 'relativity of simultaneity' effect, since, with a suitable choice of β_v it seems possible that $\Delta\mathrm{t}' = 0$ when $\Delta\mathrm{t} \neq 0$ so that two events which are simultaneous in S′ are not so in S[15]. As is clear from inspection of (8.19) and (8.20) (since $\mathrm{x}'(\mathrm{C}') = 0$, so that $\Delta\mathrm{x}'(\mathrm{C}') = 0$, which is impossible for a space-like interval) different events on the world line of the same clock always correspond

[13] For greater generality, roman symbols giving coordinate-system- and clock-synchronistion-independent equations, as introduced in Section 2 above, are used.

[14] As discussed in Ref. [47] a second, independent, invariant is associated with the relation $G_y^2 + G_z^2 = (G'_y)^2 + (G'_z)^2 =$ constant, associated with the the transformation equations (2.10) where $\phi(\pm v) = 1$.

[15] From (8.18), $G'_0 = V\Delta\mathrm{t}' = 0$ requires that $\beta_v = G_0/G_x$, so that $G_0 = V\Delta\mathrm{t} \neq 0$, if $G_x \neq 0$, and (8.17) gives: $\Delta\mathrm{x}' = G'_x = \gamma(\beta_v)(1-\beta_v^2)G_x = \Delta\mathrm{x}/\gamma(\beta_v)$, which is a LC effect inverse to that discussed in Section 4 above. However, the position-independent TD relation: $\Delta\mathrm{t} = \gamma(\beta_v)\Delta\mathrm{t}'$ which applies to each of two spatially-separated clocks, does not allow $\Delta\mathrm{t}' = 0$ for $\Delta\mathrm{t} \neq 0$ for pairs of events, one on each of the world lines of two synchronised clocks, as in Eq. (8.28) below.

to a time-like space-time interval. Events on the world lines of C'_1 or C'_2 are therefore associated with time-like intervals:

$$V^2\mathrm{t}'(\mathrm{C}'_1)^2 = V^2\mathrm{t}_1(\mathrm{C})^2 - \mathrm{x}(\mathrm{C}'_1)^2 = \left(\frac{V\mathrm{t}_1(\mathrm{C})}{\gamma(\beta_v)}\right)^2 > 0, \quad (8.24)$$

$$V^2\mathrm{t}'(\mathrm{C}'_2)^2 = V^2\mathrm{t}_2(\mathrm{C})^2 - \mathrm{x}(\mathrm{C}'_2)^2 = \left(\frac{V\mathrm{t}_2(\mathrm{C})}{\gamma(\beta_v)}\right)^2 > 0 \quad (8.25)$$

so that if $\mathrm{t}_1(\mathrm{C}) = \mathrm{t}_2(\mathrm{C})$, then $\mathrm{t}'(\mathrm{C}'_1) = \mathrm{t}'(\mathrm{C}'_2)$. Therefore simultaneous events on the world lines of two spatially-separated clocks in any inertial frame are observed to be simultaneous in any other inertial frame, contrary to what seems to be implied by the generic invariant-interval equation (8.23). To see more clearly the fallacious nature of the prediction of RS by Eq. (8.23), consider now an arbitary pair of space-like separated events, one on the world line of C'_1, the other on the world line of C'_2. It follows from (8.4)-(8.6) that:

$$(\Delta x)^2 - V^2(\Delta t)^2 = (D'_2 - D'_1)^2 - 2v(D'_2 - D'_1)[t_2(\mathrm{C}) - t_1(\mathrm{C})] + (v^2 - V^2)[t_2(\mathrm{C}) - t_1(\mathrm{C})]^2, \quad (8.26)$$

$$(\Delta x')^2 - V^2(\Delta t')^2 = (D'_2 - D'_1)^2 - V^2[t'(\mathrm{C}'_2) - t'(\mathrm{C}'_1)]^2, \quad (8.27)$$

where $\Delta x \equiv x(\mathrm{C}'_2) - x(\mathrm{C}'_1)$, $\Delta t \equiv t_2(\mathrm{C}) - t_1(\mathrm{C})$ etc[16]. Since (8.24) and (8.25) show that $t_1(\mathrm{C}) = t_1(\mathrm{C})$ implies that $t'(\mathrm{C}'_1) = t'(\mathrm{C}'_2)$, and *vice versa*, (8.26) and (8.27) give, on setting $\Delta t = \Delta t' = 0$:

$$-\Gamma^2 = (\Delta x)^2 - V^2(\Delta t)^2 = (\Delta x')^2 - V^2(\Delta t')^2 = (D'_2 - D'_1)^2 > 0 \quad (\Delta t = \Delta t' = 0). \quad (8.28)$$

There is therefore no RS effect, or correlated LC effect, associated with space-like separated events on the world lines of spatially-separated and synchronised clocks. In contrast to the spurious velocity-dependent RS and LC effects, the invariant interval relation (8.28), demonstrating the invariance of spatial separations, is valid for arbitary values of β_v.

An important difference between, on the one hand, the space-time LT of (8.19), (8.20) and the kinematical LT (8.13), and, on the other hand, the generic LT of (8.17), (8.18), is that in the former, physically relevant, transformation equations, the spatial and temporal components are not physically independent, whereas in the latter the values of G_0 and G_x may, in general, be completely arbitary. In (8.19) and (8.20) the value of $\mathrm{x}'(\mathrm{C}')$ is fixed by the initial conditions of the problem. The values of $\mathrm{x}(\mathrm{C}')$ and $\mathrm{t}'(\mathrm{C}')$ are then fixed by the transformation equations given the value of $\mathrm{t}(\mathrm{C})$. The transformation equations therefore allow only one of $x(\mathrm{C}')$, $t'(\mathrm{C}')$ and $t(\mathrm{C})$ to be chosen in an arbitary manner. In (8.12) and (8.13), the value of the spatial component U'_x is determined by that of the temporal one, U'_0 (and *vice versa*): $U'_x = \sqrt{(U'_0)^2 - 1)}$, $U'_0 = \sqrt{1 + (U'_x)^2}$. Also, for ponderable physical objects, the four vector velocity is always time-like:

$$U_0^2 - U_x^2 = (U'_0)^2 - (U'_x)^2 = 1 > 0. \quad (8.29)$$

Two previously published axiomatic derivations of the LT by the present author are now compared with that of the present paper, before a brief survey of some other derivations of the LT in the literature that do not invoke Einstein's second postulate.

[16] Note that $\Delta t = \Delta\mathrm{t}$, $\Delta x = \Delta\mathrm{x}$ etc.

The first derivation [13] was based on three explicitly stated postulates:

(A) `The LT is a single-valued function of its arguments.`

(B) `Reciprocal space-time measurements of similar rulers and clocks at rest in two different inertial frames yield identical results.`

(C) `Spatial isotropy`

Imposing the initial conditions of the problem (that the frame S′ moves with speed v along the common x,x'-axis) gives from a general multilinear function, which is necessarily single-valued, the space transformation equation:

$$x' = \gamma(x - vt), \tag{8.30}$$

where γ is an even function of v. A similar argument applied to the reciprocal experiment gives:

$$x = \gamma'(x' + v't) = \gamma(x' + vt'), \tag{8.31}$$

where postulate B, which may be called the Measurement Reciprocity Postulate (MRP) [3] is used to show that $v' = v$, $\gamma' = \gamma$ [17] Eliminating x' between (8.30) and (8.31) gives

$$t' = \gamma(t - \frac{x\eta(v)}{v}), \tag{8.32}$$

where (c.f. Eq. (3.2)) $\eta(v) \equiv (\gamma(v)^2 - 1)/\gamma(v)^2$. Differentiating (8.30) and (8.32), with respect to space-time coordinates and taking their ratio, introducing the definitions $w \equiv dx/dt$, $u \equiv dx'/dt'$ and rearranging, gives the velocity transformation relation:

$$w = \frac{u + v}{1 + \frac{u\eta(v)}{v}}. \tag{8.33}$$

Requiring that w is invariant on exchanging u and v in (8.33) gives; $u\eta(v)/v = v\eta(u)/u$ or (c.f. Eq. (3.17))

$$\frac{\eta(v)}{v^2} = \frac{\eta(u)}{u^2} = \pm\frac{1}{V^2}, \tag{8.34}$$

where V is a universal constant with the dimensions of velocity. Requiring single-valuedness of the transformation equations eliminates the solution with a minus sign in the last member of (8.34). The positive sign gives $\eta(v) = (v/V)^2$ and $\gamma(v) = 1/\sqrt{1-(v/V)^2}$. Substituting $\eta(v)$ and $\gamma(v)$ in (8.29) and (8.31), the LT is obtained. In this derivation the concepts of primary and reciprocal experiments are introduced and used, but, (as in all others in the literature of which the present author is aware) no precise operational definition is assigned to the space-time coordinate symbols. The space transformation equations (8.30) and (8.31) are correctly derived from the assumed postulates, and postulate B as applied to the TD effect does require $v' = v$ and $\gamma' = \gamma$. However, in (8.30) x and x' are the coordinates of an object at rest in S′, so that x' is constant and $x = vt +$ constant, whereas in the reciprocal experiment described by (8.31), x and x' are the *coordinates of an object at rest in* S, so that x is constant and $x' = -vt' +$ constant.

[17] This can be shown by consideration of the TD effect in the reciprocal experiment. Actually in Ref. [13] it was applied to the LC effect, the spurious nature of which, the present author was not yet aware.

But in order to derive (8.32) from (8.30) and (8.31) it is necessary to assume that x' is the *same coordinate* in these two equations. In fact, if x' in (8.30) and (8.31) is indeed the same coordinate, (that of an object at rest in the frame S$'$), then (8.31) must hold for the primary experiment, whereas it is derived for the reciprocal one. This formula was introduced in Section 2 above as the Inverse Transformation Posulate (ITP), Eq. (2.21). The time tranformation equation (8.32) is then derived, not, as claimed, from an analysis of the reciprocal experiment and application of the postulate B, but as a consequence of the tacitly assumed ITP. In fact, in Section 2, the postulate B (equality of the TD factor for the primary and reciprocal experiments) is a necessary *consequence* of the ITP. So (8.30) and (8.31), and hence (8.32), are correct, but the spatial coordinates are those of an object at rest in S$'$ with, for example, world lines $x' = 0$, $x = vt$ in S$'$, S respectively. It then follows that $dx'/dt' = 0 \neq u$ and $dx/dt = v \neq w$, so the derivation of (8.33) in Ref. [13] is fallacious in spite of the fact that the equation obtained is the correct TDVTR connecting w, u and v. The same argument mathematically invalidates Einstein's original derivation [5] of the TDVTR by effectively differentiating the space and time transformation equations and taking their ratio. In the present paper, the relation (8.34), demonstrating the existence of a limiting velocity, is obtained by consideration not of the TDVTR (8.33) but of the differential version of the time transformation equation (8.32) (c.f. Eq. (3.5)) which is used to derive the transformation law (3.20) of the TD factor γ. The derivation of Ref. [13] then effectively employs not the postulate B, but instead, tacitly, the ITP, to write down (8.31) as applied not, as claimed, to the reciprocal experiment, but instead to the primary experiment. [18] However, as shown in Section 2 above once the ITP is assumed —equivalent to setting $\delta = \gamma$ and $\Delta = 1$ in the inverse transformation equations (2.4)-(2.7)— then postulate B, the MRP, as applied to TD, necessarily follows. Explicitly, (2.26) and (2.27), which manifest the TD effects for the primary and reciprocal experiments, are a consequence of the ITP and the assumed initial conditions of the experiments.

The second derivation of the LT in Ref. [14] is based on the postulate A above and of that of Space Time Exchange Symmetry (STES):

(D) `The equations describing the laws of physics are invariant with respect to the exchange of space and time coordinates, or, more generally, with respect to the exchange of the spatial and temporal components of four-vectors.`

In order to apply STES it is necessary to introduce, *a priori*, a universal constant, V^*, with the dimensions of velocity, in order to render time and space coordinates uni-dimensional so that the STES operation may be applied in physical equations. Then $t \rightarrow x_0 \equiv V^* t$ and $t' \rightarrow x'_0 \equiv V^* t'$. Thus (8.30) is written as ($\beta \equiv v/V^*$):

$$x' = \gamma(\beta)(x - \beta x_0). \tag{8.35}$$

The STES operations $x \leftrightarrow x_0$, $x' \leftrightarrow x'_0$ applied to (8.35) give:

$$x'_0 = \gamma(\beta)(x_0 - \beta x). \tag{8.36}$$

[18] In Ref. [13] the equation equivalent to (8.31) is Eq. (2.13), and in the Appendix of Ref. [3] where the calculations of Ref. [13] are reviewed, it is Eq. (A.22)

Eliminating x between these two equations gives:

$$x_0 = \frac{1}{\gamma(\beta)(1-\beta^2)}(x'_0 - \beta x'). \tag{8.37}$$

Applying the ITP to the time transformation equation (8.36) gives:

$$x_0 = \gamma(\beta)(x'_0 + \beta x'). \tag{8.38}$$

Consistency of (8.38) and (8.37) requires that

$$\gamma(\beta) = \frac{1}{\sqrt{1-\beta^2}} \tag{8.39}$$

so that (8.35) and (8.36) are the usual LT equations. Since (8.39) requires that the maximum value of v is V^*, it follows that $V^* \equiv V$, where V is the maximum relative relative velocity of two inertial frames as introduced in Section 3 above as the solution of the equation $\eta(V) = \pm 1$. In fact, in Ref. [14] the inverse transformation was written as

$$x_0 = \gamma'(\beta')(x'_0 + \beta' x'), \tag{8.40}$$

which, for consistency with (8.37), requires

$$\gamma'(\beta') = \frac{1}{\gamma(\beta)(1-\beta^2)}, \quad \beta = \beta', \quad \gamma(\beta) = \gamma'(\beta'). \tag{8.41}$$

An interesting feature of this derivation, apart from its extreme brevity, is the absence of any 'V^2 sign ambiguity'.

It can be seen that this derivation also (tacitly) assumes the ITP (equivalent to $\delta = \gamma$, $\Delta = 1$) in writing down (8.40), which was done without any justifying comment in Ref. [14]. Comparing (8.36) with the corresponding general transformation equation (2.2) shows the that the STES operation is equivalent to setting $\delta = \gamma$ and $\omega = -\beta\gamma/V$. Subsituting these values as well as $\rho = -\gamma v$, given by the initial conditions, and $\Delta = 1$, given by the ITP, into Eq. (2.7) it is found that

$$\Delta = \gamma\delta - \omega\rho = 1 = \gamma^2 - \beta^2\gamma^2 \tag{8.42}$$

from which (8.39) follows directly.

The early derivations of the space-time LT by Ignatowsky [17], Frank and Rohe [18] and Pars [19], without explicit consideration of light signals or classical electromagnetism, will now be briefly reviewed and compared with that of the present paper. For clarity, the same notation for the coefficients in the space-time transformation equations as employed above will be used.

It has been claimed [48] that the first 'lightless' derivation of the LT is that due to Ignatowsky [17], but closer examination reveals an essential logical hiatus in this derivation. After introducing derivatives of the space and time coordinates in the LT, written as in (2.8) and (2.9), (but without specifying any operational meaning of these coordinates), applying the the postulate B (MRP) to length intervals in the frames S and S′, and tacitly setting $\Delta = 1$, Ignatowsky derives the velocity transformation equation

$$u = \frac{w - v}{1 - \frac{\eta(v)}{v^2} wv} \tag{8.43}$$

that is a transposition of (8.33) above. Considering the series of inertial frames S′, S″, S‴, ... moving with speeds v, v', v'',..., relative to S, Ignatowsky then asserts that:

$$\frac{\eta(v)}{v^2} = \frac{\eta(v')}{v'^2} = \frac{\eta(v'')}{v''^2} = ... = n, \tag{8.44}$$

where n is a universal constant with dimensions (velocity)$^{-2}$. Setting $n = 1/c^2$ where c is the velocity of light in free space, then makes (8.43) identical to the TDVTR derived previously by Einstein [5] and Poincaré [49]. But no supporting argument is given for the ansatz (8.44). In order to derive it, some symmetry properties of (8.43) must be postulated. For example demanding that u is invariant under the substitutions $w \to -v$, $v \to -w$ gives, since $\eta(v)$ is an even function of v,

$$\frac{\eta(v)}{v^2} = \frac{\eta(w)}{w^2} = n, \tag{8.45}$$

(c.f. Eqs. (3.17),(8.34)), where n is Ignatowsky's universal constant, since v and w are independent. This is a proof of Ignatowsky's anstatz (8.44) that was missing in order to derive from it the TDVTR and hence to obtain the LT.

Frank and Rothe [18] start from equations equivalent to (2.11) and (2.12), without specifying the physical meaning of the space and time coordinate symbols in the equations, and tacitly invoke the ITP to write down an equation equivalent to (2.22). Eliminating x' between the equations equivalent to (2.11) and (2.22) gives an equation equivalent to (8.32) above. Taking the ratio of the original space transformation equation to the so-obtained time transformation equation and introducing the definitions $u \equiv x'/t'$ and $w \equiv x/t$ gives Eq. (8.43) above. In this equation, as conventionally interpreted, w is the speed of an object in the frame S, and u is the speed of the same object in the frame S′. If there exists a limiting velocity, V, which is the maximum speed of an object in any inertial frame, then, on the assumption that an object attains this velocity simultanously in the frames S and S′ when $u = w = V$, (8.43) Frank and Rothe remark that (8.43) must have the solution:

$$V = \frac{V - v}{1 - \frac{\eta(v)V}{v}} \tag{8.46}$$

or

$$1 = \frac{1 - \beta}{1 - \frac{\eta(v)}{\beta}}. \tag{8.47}$$

Rearranging this equation gives $\eta(v) \equiv (\gamma^2 - 1)/\gamma^2 = \beta^2$, so that γ is given by Eq. (8.39), and the LT is derived. Since Frank and Rothe set $V = c$ they are essentially demanding that the velocity transformation formula (8.43) be consistent with Einstein's second postulate, which sheds doubt on the claims to be found in the literature that this derivation does not invoke the second postulate. In any case, in the primary experiment considered by Ignatowsky it is not possible to set $x/t = w \neq v$, or x', which is constant, equal to ut' and in the reciprocal experiment considered by Frank and Rothe $x'/t' = -v \neq u$ and $x =$ constant $\neq wt$. The derivation of the TDVTR (8.43) by both Ignatowsky and Frank and Rothe (as in Einstein's original derivation, and that of the present author in Ref. [13]) is therefore mathematically unsound. Also in (8.43), Frank and Rothe erroneously assume that the TDVTR, not the RRVTR, is the correct transformation formula for velocities observed in the same space-time experiment. It is demonstrated above that this interpretation of the TDVTR is excluded by the results of the Hafele and Keating experiment.

The first mathematically correct, published, derivation of the LT that did not invoke Einstein's second postulate, known to the present author, is that due to Pars [19]. There are several similar subsequent derivations in the literature [50, 51, 29, 52] which also assume that the space-time transformation equations satisfy a group property. Three inertial frames, S, S′ and S″ are considered. The relative velocities of (S, S′), (S′, S″) and (S, S″) in the corresponding space-time transformation equations are: v, u and w respectively. This gives, writing the unknown coefficients in the transformation equations as in Eqs. (2.8) and (2.9):

$$
\begin{aligned}
x' &= \gamma(v)(x - vt), && (8.48)\\
t' &= \omega(v)x + \delta(v)t, && (8.49)\\
x'' &= \gamma(u)(x' - ut') = \gamma(w)(x - wt), && (8.50)\\
t'' &= \omega(u)x' + \delta(u)t' = \omega(w)x + \delta(w)t. && (8.51)
\end{aligned}
$$

Self-consistency of these equations, (i.e. substituting x' and t' given by (8.48) and (8.49) in (8.50) and equating coefficients of t and x respectively) gives:

$$
\begin{aligned}
v\gamma(v)\gamma(u) + u\gamma(u)\delta(v) &= w\gamma(w), && (8.52)\\
\gamma(v)\gamma(u) - u\gamma(u)\omega(v) &= \gamma(w). && (8.53)
\end{aligned}
$$

Since $w = 0$ when $u = -v$, and both γ and δ are even functions of their arguments, (8.52) implies

$$\gamma(v) = \delta(v). \tag{8.54}$$

Interchanging u and v in (8.53), assuming that this leaves w invariant, and combining the equation so-obtained with (8.53) gives:

$$u\gamma(u)\omega(v) = v\gamma(v)\omega(u) \tag{8.55}$$

so that

$$\frac{u\gamma(u)}{\omega(u)} = \frac{v\gamma(v)}{\omega(v)} = -(V^{\dagger})^2, \tag{8.56}$$

where $V^{\dagger}$ is a universal constant with dimensions of velocity, since u and v are independent variables. Using (8.56) to eliminate $\omega(v)$ from (8.53) gives

$$\gamma(w) = \gamma(v)\gamma(u)\left[1 + \frac{uv}{(V^{\dagger})^2}\right]. \tag{8.57}$$

Setting $u = -v$, $w = 0$ in this equation gives, since $\gamma(0) = 1$,

$$\gamma(v) = \frac{1}{\sqrt{1 - \left(\frac{v}{V^{\dagger}}\right)^2}}. \tag{8.58}$$

Showing from (3.18) that $V^{\dagger} \equiv V$ where V is the limiting velocity introduced in Section 2 above. The LT is then obtained on making the substitutions $\delta(v) = \gamma(v)$ and $\omega(v) = -v\gamma(v)/V^2$ in (8.48) and (8.49). Given (8.54) and (8.56), (8.58) implies that $\Delta = 1$:

$$\Delta \equiv \gamma(v)\delta(v) - \omega(v)\rho(v) = \gamma(v)^2 + v\omega\gamma(v) = \gamma(v)^2\left[1 - \left(\frac{v}{V^{\dagger}}\right)^2\right] = 1. \tag{8.59}$$

In this derivation, as in that of Ref. [13] and that presented in the present paper, the necessary existence of the limiting velocity $V^{\dagger} \equiv V$ is established naturally in the course of the derivation (compare (8.56) with (3.17)). Since $\delta(v) = \gamma(v)$ the ratio of (8.52) to (8.57) gives the TDVTR of (8.33), and (8.34) with $+1/V^2$ on the right side. Assuming that the transformation equations form a group then leads to a mathematically sound derivation of both the LT and the TDVTR,but does not address the operational meaning of these equations in physical applications. As discussed in Section 5 above, the RRVTR that actually describes velocity transformations in the same space-time experiment, does not satisfy the group property, and so evidently cannot be derived by introducing the latter as the sole postulate in order to construct the theory. Also, Pars did not give any reason for not considering a possible solution with $+(V^{\dagger})^2$ on the right side of (8.56), i.e. the sign ambiguity problem was not addressed.

To summarise the above discussion of various axiomatic derivations of the LT: Both of the previously published derivations by the present author, the first [13], for which the essential axiom (replacing Einstein's Special Relativity Principle) was claimed to be the the Measurement Reciprocity Postulate, B, and the second [14], based on Space Time Exchange Symmetry (postulate D) tacitly assumed, and used in an essential way, the Inverse Transformation Postulate introduced in Section 2 above (equivalent to setting $\delta = \gamma$ and $\Delta = 1$ in the inverse transformation equations (2.4)-(2.7)). The ITP was essential for the second derivation, and actually rendered redundant the MRP in the first derivation. This is because, as shown in Eqs. (2.26) and (2.27), the MRP, as applied to the TD effect, is a necessary consequence of the ITP; this is evident even at the stage of the derivation where the TD factor γ is still an unknown even function of v.

Since the derivations of both Ignatowsky and Frank and Rothe use a time transformation equation equivalent to (3.1) or (8.32) in to derive the TDVTR, these authors also both tacitly invoke the ITP in their derivations. In the first published derivation of the present author and in those of both Ignatowsky and Frank and Rothe, the derivations of the TDVTR (though obtaining the correct result) are mathematically flawed since it is impossible in the primary experiment, considered by Ignatowsky, that $dx/dt = w \neq v$, or in the reciprocal experiment, considered by Frank and Rothe, that that $dx'/dt' = u \neq -v$. For the correct derivation of the transformation law of γ (algebraically equivalent to the TDVTR) a third inertial frame S$''$ must be explictly introduced, as in the derivation of (3.20) above, starting from the time transformation equation (3.1).

The derivation of Pars [19], in which the group property is invoked, is mathematically sound (the space-time coordinates are those of the same physical object in all equations) since a third inertial frame S$''$ (the proper frame of the object considered) is explicitly introduced. In this way a correct derivation of the TDVTR is obtained as an intermediate step in the derivation of the LT. In this case, there is no use (tacit or otherwise) of the ITP, since inverse transformations or reciprocal experiments are not even considered. The relation $\delta = \gamma$ and the existence of a limiting velocity follow directly from the self-consistency conditions, (8.52) and (8.53), furnished by the postulated group property of the transformations. The other consequence of the ITP, $\Delta = 1$, also follows from the self-consistency conditions (Eq. (8.59)). Although the group-property-based derivations of Pars and later authors do establish, in a mathematically sound way, the existence of a limiting velocity, the TDVTR, and the LT, their physical interpretation, in particular the different operational meanings of the TDVTR and RRVTR velocity transformation equations, explained in Sections 5 and 6 above, is not addressed.

Where does all this leave our fundamental understanding of the space-time physics of inertial frames? Neither the present paper nor any of the others just discussed have invoked Einstein's first postulate, the Special Relativity Principle (SRP) which was already known to, and stated by, Galileo and Newton:

(SRP) `The dynamical laws of physics are the same in all inertial frames.`

All of space-time and kinematical special relativity physics is a consequence of the LT —actually of only the TD relation (8.3) and the transformation law of the TD factor, γ, Eq. (8.9)— which are physically equivalent to space-time and kinematical LT's respectively. But, given the general transformation equations (2.1)-(2.3) and their inverses (2.4)-(2.7) as required by single-valuedness or space-time homogeneity, extremely simple and purely mathematical postulates unrelated to any dynamical theory, as cited in the SRP, suffice to derive (8.3) and (8.9) as well as the nececesary existence of the limiting velocity V. As demonstrated above, it is sufficient to assume the ITP, or, equivalently to set $\delta = \gamma$, $\Delta = 1$ in (2.1)-(2.7). Alternatively, as first demonstrated by Pars, it is enough to postulate that the transformation equations respect the group property. Introducing, *a priori*, a constant velocity to enable exchange of unidimensional, [L], space and time coordinates in physical equations, and postulating that the system of space and time transformation equations rests invariant under exchange of these unidimensional coordinates leads, together with the ITP, to an alternative, very short, derivation of the LT without any sign ambiguity related to the limiting velocity [14].

Such purely mathematical considerations do not however reveal the physical meaning (i.e the relevance to experimental measurements) of the space and time coordinates, velocities, momenta and energies that appear in the transformation equations. For this careful consideration of the relation of the mathematical symbols which appear in the equations to measurements, particularly of the space and time intervals from which velocities and other kinematical quantities are derived, is needed. As seen in the discussion above, of primordial importance for this is the correct specification of the initial conditions of the space-time experiments considered, in particular the essential concepts of physically-independent primary and reciprocal experiments and of and base and travelling frames, as discussed above in Sections 2 and 6 respectively. It is insuffient attention paid to these fundamental physical concepts that is at the origin of the many paradoxes and physical misinterpretations of conventional special relativity theory, resulting from purely mathematical manipulation of the space-time coordinates of the standard LT, without due regard to the operational meaning of the these symbols.

Einstein's exact statement of the SRP was [5]:

`'The laws by which the states of physical systems undergo change are not affected, whether these changes of state are referred to one or the other of two systems in uniform translatory motion.'`

This makes explicit that the SRP is a statement about dynamics: 'The laws by which the states of physical systems undergo change...', whereas in all the discussions of space-time and kinematical transformations above there is nowhere any mention of dynamical laws. The only laws discussed which respect the SRP are the purely kinematical ones of conservation of energy and momentum and of the invariance of effective mass in collision processes where particles may be created and destroyed. These laws are separately valid in each inertial frame. However, the kinematical transformation laws of the energy-momentum four vector do not describe observations in different inertial frames in the same space-time experiment but relate instead inertial frame configurations in *different* space-time experiments. The laws for kinematical (e.g. velocity) transformations between two inertial frames in the same space-time experiment are different for a base frame transformation into a travelling frame (Eq. (5.11)) and for a travelling frame transformation into the base frame (Eq. (5.12)). These, and other examples given above, show that what at first seems an obvious corollory of the SRP:

`Space-time coordinates in different inertial frames give equivalent descriptions of nature, according to the same laws.`

does not hold true. In simple space-time experiments where clocks at rest and in motion are compared, there is a clear difference for observers in the base and travelling frames. For the former the moving clock runs slow, for the latter it runs fast. In the HK experiment, although the velocities of the aircraft relative to the surface of the Earth are the same in the W−E and E−W flights, in the former the airborne clocks run slower, in the latter faster, than a clock at rest on the surface of the Earth.

The mathematical expression of the SRP is the covariance of physical laws. The most notable example of this is Maxwell's Equations, discussed at length in Einstein's seminal paper that was actually entitled: 'On the Electrodynamics of Moving Bodies'. However, as demonstrated above, it is possible to establish all of the space-time geometry and kinematics of special relativity without consideration of Classical Electro-Magnetism (CEM) or any other dynamical theory.

Recently published work by the present author [6] has shown that all the dynamical equations of CEM (Maxwell's Equations, the Lorentz Force Equation) can be derived by consideration of the classical limit of Quantum Electro-Dynamics (QED). The fundamental quantum process underlying inter-charge forces is Møller scattering: ee $\rightarrow$ ee. In the resulting theory, Relativistic Classical Electro-Dynamics (RCED), the equations of Faraday and Maxwell's CEM are recovered as the approximation valid at the lowest relevant order in β. As remarked by Einstein, magnetic forces are a consequence of special relativity and vanish in the Galilean ($c \rightarrow \infty$) limit and it is found that the equations of CEM are recovered from those of RCED by neglecting terms of order higher than β^2. However RCED predicts higher order corrections to Maxwell's Equations that break their covariance. These occur, in lowest order, at β^4 for the electric field divergence equation, and at β^5 for Ampère's Law [53]. A corollory is the breakdown at $O(\beta^4)$ of the Gauss law for the electric field of a uniformly moving charge [54]. The covariance of Maxwell's Equations, which originally lead Lorentz to propose the transformation named for him, are therefore only an approximation of the relativistic theory, valid at $O(\beta^2)$. RCED is therefore a dynamical theory that does not, except at $O(\beta^2)$, respect the SRP. In this case the rest frame of the source charge constitutes a preferred frame.

Another prediction of QED is the instantaneous nature of inter-charge forces in their CM system of the interacting charges [6]. This is a consequence of the space-like nature of the exchanged virtual photons that that transmit, in QED, the inter-charge force. Since in Møller scattering all particles have the same mass, then the energies in the CM system of the incoming and scattered electron are the same (see Eq. (7.28)). However, a non-zero scattering angle, necessary to transmit an electrodynamic force, implies that the virtual photon has a non-vanishing momentum. Then, since the energy of the photon vanishes, but not its momentum, its energy-momentum four vector is space-like (c.f. Eq. (8.21)) and, according to Eq. (7.5), its velocity is infinite, corresponding to an instantantaneous inter-charge force. The same conclusion is reached by an independent argument based on consideration of the Fourier transform of the momentum-space invariant Møller scattering amplitude [6]. Superluminal propagation speeds of magnetic force fields were aleady clearly seen by Hertz, at small source-detector separations, in the original series of experiments in which 'electromagnetic waves' (due, in QED, to the propagation of real photons with light-like four vectors) were first observed at larger distances [55, 56]. A recent experiment [20] has given convincing evidence for superluminal propagation of magnetic force fields, in agreement with the QED prediction [21]. The frame velocity of a space-like virtual photon, unlike that of a ponderable physical object with a time-like four vector, is therefore observed to be not limited to V, in agreement with the prediction of relativistic kinematics.

The final conclusions of the present work are: Einstein's first postulate of special relativity, the dynamical Special Relativity Principle, has no relevance for the simple axiomatic derivation presented here of the essential equations of special relativity, (8.3) (time dilation) and (8.9) (transformation law of the time dilation factor γ) based, apart from single-valuedness or spatial homogeneity and spatial isotropy, on the purely mathematical Inverse Transformation Postulate ($\delta = \gamma$ and $\Delta = 1$ in Eqs. (2.4)-(2.7)). These equations, together with the Galilean world lines of the object considered, have the same physical content as the space-time and kinematical Lorentz Transformations, when the latter are correctly interpreted. The derivation also demonstrates the necessary existence of a universal upper limit, V, on the relative velocity of two inertial frames. The ITP was actually essential for several previously published derivations, though not explicity stated, because it seems 'obvious' and so was assumed to hold without any justifying argument. Einstein's second postulate, concerning the constancy of the speed of light, c, follows, as a very good approximation, from relativistic kinematics: $c = V$, where V is the maximum relative velocity of two inertial frames, if the energy equivalent of the mass of the photon is very small compared to its energy. However, the observed existence of photons in the same reference frame with a wide range of energies is incompatible with the existence of strictly massless photons, if relativistic kinematics is valid, showing, in these circumstances, that Einstein's second postulate is only an (albeit excellent) approximation.

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