The Hafele-Keating experiment, velocity and length interval transformations and resolution of the Ehrenfest paradox

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Abstract

A relativistic analysis based on the paths, in a non-rotating frame comoving with the centroid of the Earth, of clocks carried by aircraft circumnavigating the Earth in different directions, as in the Hafele-Keating experiment, predicts time differences between airborne and Earth-bound clocks at variance with the results of the experiment. The latter imply new relativistic velocity transformations differing from the conventional ones. These transformations demonstrate in turn the invariance of length intervals on the surface of the rotating Earth and so resolve the Ehrenfest paradox for this case.

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In the Hafele-Keating (HK) experiment [1], performed in 1971, four caesium-beam atomic clocks were flown around the world in commercial aircraft, once in the west-to-east (W$-$E) and once in the east-to-west (E$-$W) directions. The time intervals recorded by the clocks during the flights were compared with those recorded by reference clocks at the U.S. Naval Observatory. The time intervals for the airborne clocks were sensitive to both special relativitistic (SR) and general relativistic (GR), or gravitational, effects. Here only SR effects are considered. It is assumed, for simplicity, in the following calculations, that the clock C$''$ at rest in the comoving frame S" of the aircraft executes an equatorial circumnavigation of the Earth at constant speed v'_A in the comoving frame S' of the ground-based clock C$'$. The latter moves with constant speed $v_E = \Omega R$ relative to a non-rotating inertial frame S comoving with the centroid of the Earth; that is, the rotation of the Earth around the Sun is neglected. The parameter Ω is the angular frequency of rotation of the Earth and R is its equatorial radius. Since gravitational effects are neglected, the altitude of the aircraft during the flights may be neglected in comparision with R.

The clocks C$'$ and C$''$ undergo transverse acceleration due to the rotation of the Earth, but experiments with decaying muons in near-circular orbits in a storage ring at CERN demonstrated that the special-relativistic time dilation (TD) effect is the same as for uniform motion at the same speed, v, where $v/c = 0.9994$ or $\gamma = \sqrt{1-(v/c)^2} = 29$, with a relative precision of 0.1 % in the presence of a transverse acceleratation, due to the bending field of the storage ring, of $10^{18}g$ [2]. The time dilation and other relativistic effects in the HK experiment can therefore be calculated with confidence on the assumption that S" and S' are inertial frames moving with speeds v'_A and v_E relative to the frames S' and S respectively.

For the analysis of the experiment a hypothetical clock, C, at rest in the frame S, registering 'coordinate time' is introduced [3, 4]. If T' is the time interval recorded by the Earth-bound clock during either the W−E or the E−W flights, then

$$T' = \frac{2\pi R}{v'_{\rm A}}. \tag{1}$$

If the distances and times travelled by the aircraft in the frame S during the round trips are denoted by d, T respectively (where T denotes an unobserved coordinate time interval registered by C) then:

$$d({\rm W-E}) = v_{\rm E}T({\rm W-E}) + 2\pi R, \tag{2}$$

$$d({\rm E-W}) = v_{\rm E}T({\rm E-W}) - 2\pi R. \tag{3}$$

Using the conventional relativistic parallel velocity addition relations to give the velocity $\hat{v}_{\rm A}$ of the aircraft in the frame S:

$$\hat{v}_{\rm A}({\rm W-E}) = \frac{v_{\rm E} + v'_{\rm A}}{1 + \frac{v_{\rm E}v'_{\rm A}}{c^2}}, \tag{4}$$

$$\hat{v}_{\rm A}({\rm E-W} = \frac{v_{\rm E} - v'_{\rm A}}{1 - \frac{v_{\rm E}v'_{\rm A}}{c^2}} \tag{5}$$

(2) and (3) give, on using Eq. (1) to eliminate $2\pi R$, the flight times in the frame S:

$$\hat{T}({\rm W-E}) \equiv \frac{d({\rm W-E})}{\hat{v}_{\rm A}({\rm W-E})} = T'\gamma({\rm C}')^2(1 + \frac{v_{\rm E}v'_{\rm A}}{c^2}), \tag{6}$$

$$\hat{T}({\rm E-W}) \equiv \frac{d({\rm E-W})}{\hat{v}_{\rm A}({\rm E-W})} = T'\gamma({\rm C}')^2(1 - \frac{v_{\rm E}v'_{\rm A}}{c^2}) \tag{7}$$

where $\gamma({\rm C}') \equiv 1/\sqrt{1-(v_{\rm E}/c)^2}$.

The TD effect between the frames S and S" is:

$$\Delta t = \gamma({\rm C}'')\Delta t'' \tag{8}$$

where

$$\gamma({\rm C}'') = \gamma'({\rm C}'')\gamma({\rm C}')(1 \pm \frac{v_{\rm E}v'_{\rm A}}{c^2}) \tag{9}$$

and $\gamma'({\rm C}'') \equiv 1/\sqrt{1-(v'_{\rm A}/c)^2}$. In Eq. (9), the $+(-)$ signs correspond to the W−E (E−W) flights.

The time difference observed for the W−E flight is

$$\Delta T'({\rm W-E}) \equiv T''({\rm W-E}) - T' = T'\left(\frac{T''({\rm W-E})}{T'} - 1\right). \tag{10}$$

Setting $\Delta t = \hat{T}({\rm W-E})$ and $\Delta t'' = T''({\rm W-E})$ in (8) and eliminating the unmeasured coordinate time interval $\hat{T}({\rm W-E})$ between the resulting equation and (6) gives:

$$\frac{T''({\rm W-E})}{T'} = \frac{\gamma({\rm C}')^2}{\gamma({\rm C}'')}(1 + \frac{v_{\rm E}v'_{\rm A}}{c^2}) = \frac{\gamma({\rm C}')}{\gamma'({\rm C}'')} \tag{11}$$

where in the last member (9) has been used to eliminate $\gamma(\mathrm{C}'')$. A similar calculation for the E−W flight shows that

$$\frac{T''(\mathrm{E-W})}{T'} = \frac{\gamma(\mathrm{C}')}{\gamma'(\mathrm{C}'')} = \frac{T''(\mathrm{W-E})}{T'}. \tag{12}$$

Therefore Eqs. (10)-(12) give:

$$\Delta\hat{T}'(\mathrm{W-E}) = \Delta\hat{T}'(\mathrm{E-W}). = T'\left(\frac{\gamma(\mathrm{C}')}{\gamma'(\mathrm{C}'')} - 1\right) \tag{13}$$

Equal time differences are therefore predicted for the W−E and the E−W flights. The actual parameters of the HK experiment are well approximated by the constant values; $v'_{\mathrm{A}} = 300\mathrm{m/s}$, $v_{\mathrm{E}} = \Omega R = 470\mathrm{m/s}$ and $T' = 27\mathrm{h}$, for which (13) predicts:

$$\Delta\hat{T}'(\mathrm{W-E}) = \Delta\hat{T}'(\mathrm{E-W}) = 49\mathrm{ns}$$

Which may be compared with the predictions for the special-relativistic (SR) effect in the actual HK experiment derived by properly taking into account the actual paths followed by the aircraft over the Earth's surface as well as the time-dependence of their speeds [1]:

$$\Delta T'_{\mathrm{HK}}(\mathrm{W-E})_{\mathrm{SR}} = -184 \pm 18 \text{ ns}$$

$$\Delta T'_{\mathrm{HK}}(\mathrm{E-W})_{\mathrm{SR}} = 90 \pm 10 \text{ ns}$$

Including GR effects the overall prediction for the time differences was [1]

$$\Delta T'_{\mathrm{HK}}(\mathrm{W-E})_{\mathrm{SR+GR}} = -40 \pm 23 \text{ ns}$$

$$\Delta T'_{\mathrm{HK}}(\mathrm{E-W})_{\mathrm{SR+GR}} = 275 \pm 21 \text{ ns}$$

which were found to be in good agreement with experimentally measured values [1]:

$$\Delta T'_{\mathrm{HK}}(\mathrm{W-E})_{\mathrm{meas}} = -59 \pm 10 \text{ ns}$$

$$\Delta T'_{\mathrm{HK}}(\mathrm{E-W})_{\mathrm{meas}} = 273 \pm 7 \text{ ns}$$

Calculating instead the overall prediction by replacing the SR predictions of Ref. [1] with those given by Eq. (13) gives

$$\Delta\hat{T}'_{\mathrm{HK}}(\mathrm{W-E})_{\mathrm{SR+GR}} = 193 \text{ ns}$$

$$\Delta\hat{T}'_{\mathrm{HK}}(\mathrm{E-W})_{\mathrm{SR+GR}} = 134 \text{ ns}$$

which are evidently completely incompatible with the measured values.

It is then clear that the calculation above, based on Eqs. (2)-(5), does not describe correctly the results of the HK experiment. On reflection it is quickly seen that the mistake resides not in the purely geometrical formulae (2) and (3), the TD relation (8) or the transformation law (9), but in the velocity transformation formulae (4) and (5). These predict, when inserted in (2) and (3), that $\hat{T}(\mathrm{W-E}) \neq \hat{T}(\mathrm{E-W})$. Suppose that the aircraft start out at the same instant, and travel with the same speed relative to the surface of the Earth during the W−E and E−W flights. They will arrive back simultaneously at their starting point. There is thus a triple world line coincidence event (those of the two aircaft and the starting point on the Earth) at arrival. This must be observed as such in all frames, including S. It is therefore impossible, by this 'zeroth theorem of space-time physics', the importance of which has previously been stressed by Langevin [5, 6] and Mermin [7], that $T(\mathrm{W}-\mathrm{E}) \neq T(\mathrm{E}-\mathrm{W})$. The velocity transformation formulae (4) and (5) are therefore inapplicable to the analysis of the HK experiment in the way shown above.

Indeed, the correct SR prediction for the HK experiment can be obtained without any consideration of the the distances $d(\mathrm{W}-\mathrm{E})$ and $d(\mathrm{E}-\mathrm{W})$ covered by the aircraft in the frame S during the round trips. The TD effect between the frames S and S' is given by the relation

$$\Delta t = \gamma(\mathrm{C}')\Delta t' \tag{14}$$

from which it is clear (contrary to Eqs.(6) and (7)) that $T(\mathrm{W}-\mathrm{E}) = T(\mathrm{E}-\mathrm{W})$. It is then found by combining Eqs. (8),(9),(10) and (14) that

$$\Delta T'(\mathrm{W}-\mathrm{E}) = T'\left(\frac{\gamma(\mathrm{C}')}{\gamma(\mathrm{C}'')} - 1\right) = T'\left(\frac{1}{\gamma'(\mathrm{C}'')\left[1+\frac{v_{\mathrm{E}}v'_{\mathrm{A}}}{c^2}\right]} - 1\right). \tag{15}$$

Retaining only $\mathrm{O}(\beta^2)$ terms in (15) and the corresponding formula for $\Delta T'(\mathrm{E}-\mathrm{W})$ gives

$$\Delta T'(\mathrm{W}-\mathrm{E}) = -\frac{T'\beta'_{\mathrm{A}}}{2}(\beta'_{\mathrm{A}} + 2\beta_{\mathrm{E}}), \tag{16}$$

$$\Delta T'(\mathrm{E}-\mathrm{W}) = \frac{T'\beta'_{\mathrm{A}}}{2}(-\beta'_{\mathrm{A}} + 2\beta_{\mathrm{E}}) \tag{17}$$

where $\beta'_{\mathrm{A}} \equiv v'_{\mathrm{A}}/c$, $\beta_{\mathrm{E}} \equiv v_{\mathrm{E}}/c$. Substituting the numerical values of v'_{A} and v_{E} quoted above in (16) and (17) gives

$$\Delta T'_{\mathrm{HK}}(\mathrm{W}-\mathrm{E})_{\mathrm{SR}} = -201 \text{ ns}$$

$$\Delta T'_{\mathrm{HK}}(\mathrm{E}-\mathrm{W})_{\mathrm{SR}} = 104 \text{ ns}$$

in good agreement with the calculated predictions for the HK experiment and consistent with its results.

Setting $T(\mathrm{W}-\mathrm{E}) = T(\mathrm{E}-\mathrm{W}) = T = \Delta t$, $T' = \Delta t'$ and using Eqs. (1),(2),(3) and (14), the correct velocity transformation formulae, between the frames S' and S, for the aircraft are

$$v_{\mathrm{A}}(\mathrm{W}-\mathrm{E}) \equiv \frac{d(\mathrm{W}-\mathrm{E})}{T} = v_{\mathrm{E}} + \frac{v'_{\mathrm{A}}}{\gamma(\mathrm{C}')}, \tag{18}$$

$$v_{\mathrm{A}}(\mathrm{E}-\mathrm{W}) \equiv \frac{d(\mathrm{E}-\mathrm{W})}{T} = v_{\mathrm{E}} - \frac{v'_{\mathrm{A}}}{\gamma(\mathrm{C}')}. \tag{19}$$

These transformation formulae for relative velocities between different inertial frames in the same space-time experiment were previously derived by the present author [8, 9].

Consider now the distance, $\Delta s'(\mathrm{W}-\mathrm{E})$, in the frame S', of the aircraft from its starting point after a time interval, $\Delta t'$, sufficiently short that the curvature of the surface of the Earth may be neglected. Denoting by $\Delta d(\mathrm{W}-\mathrm{E})$ and Δt the corresponding distance moved and time interval, in the frame S, then (18) and (14) give:

$$
\begin{aligned}
\Delta d(\mathrm{W}-\mathrm{E}) & = v_{\mathrm{A}}\Delta t = v_{\mathrm{E}}\Delta t + \frac{v'_{\mathrm{A}}\Delta t}{\gamma(\mathrm{C}')} \\
& = v_{\mathrm{E}}\Delta t + v'_{\mathrm{E}}\Delta t' \\
& = v_{\mathrm{E}}\Delta t + \Delta s'(\mathrm{W}-\mathrm{E}). \qquad (20)
\end{aligned}
$$

Denoting by $\Delta s(\mathrm{W}-\mathrm{E})$ the distance in S between the aircraft and the starting point of its flight after the time interval Δt, transposition of (20) gives:

$$
\Delta s(\mathrm{W}-\mathrm{E}) = \Delta d(\mathrm{W}-\mathrm{E}) - v_{\mathrm{E}}\Delta t = \Delta s'(\mathrm{W}-\mathrm{E}). \qquad (21)
$$

The length interval between the aircraft and its starting point is therefore the same in the frames S and S' in relative motion —there is no 'length contraction' effect.

This demonstration resolves the Ehrenfest paradox [10] concerning the ratio of the circumference to the radius of a rotating disc. This ratio is simply 2π. Contrary to Einstein's assertions [11, 12], the ratio is not greater than 2π and no introduction of non-Euclidean spatial geometry is necessary.

The conventional relativistic transformation formulae (4) and (5) are not incorrect, but only misinterpreted in the calculation above. These formulae do correctly describe a kinematical transformation between configurations of two different and *physically independent* space time experiments [8, 9], not velocities as observed in different frames of *the same* space time experiment, as assumed above. In fact, writing $\hat{\gamma}(\mathrm{C}'') = 1/\sqrt{1-(\hat{v}_{\mathrm{A}}/c)^2}$ it may be shown that $\hat{\gamma}(\mathrm{C}'') = \gamma(\mathrm{C}'')$ so that the velocity transformation equation (4) is algebraically equvalent to Eq. (9), which is the transformation equation of the TD factor γ for the clock C'' between the frames S' and S.

In conclusion, the experimental results of the HK experiment falsify the conventional interpretation of the relativistic velocity transformation formulae (4) and (5), since the latter, when used to calculate flight times in the frame S predict equal time differences between the airborne and Earth-bound clocks for the W$-$E and E$-$W flights. The necessary equality of the S-frame durations of the W$-$E and E$-$W flights (in contradiction with the predictions, (6) and (7), of (4) and (5) respectively) requires the velocity transformation formulae for the HK experiment, between the frames S' and S, to be (18) and (19). These equations show further that there is no 'length contraction' effect for spatial intervals on the surface of the Earth and so resolve the corresponding Eherenfest paradox for the radius and equatorial circumference of the rotating Earth. How the spurious and correlated 'length contraction' and 'relativity of simultaneity' effects of conventional special relativity arise from a general and fundamental misinterpretation of the space-time Lorentz transformation is explained elsewhere [13, 14, 15].

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