Feynman's dynamical route to special relativity via work-to-energy conversion and Newton's Second Law

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Abstract

Adapting a derivation due to Feynman, expressions for the relativistic energy and momentum of a ponderable object are obtained from work-energy equivalence and Newton's Second Law of mechanics. Transformation laws of relativistic energy and momentum are found and the time dilation relation and the Lorentz transformation of time intervals are obtained from the invariant relation connecting the Newtonian mass of an object to its relativistic energy and momentum. Lorentz transformations of space-time events are then derived and discussed. Finally, space-time geometrical and kinematical velocity transformation formulas are compared. Although identical in Galililean relativity, they are found to differ at order $(v/c)^2$ in special relativity.

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1 Introduction

The origins of special relativity theory (SRT) are to be found in considerations related to classical electromagnetic theory. A system of transformation equations where time intervals are predicted to have different values for observers in different inertial frames, was proposed by Voigt [1] in 1887 in order to render covariant the wave equation. These transformations were adapted by Lorentz [2] in order to obtain the eponymous space-time transformation equations that are the basis of SRT. As demonstrated by Lorentz [2], as well as earlier by Larmor [3] and slightly later by Poincaré [4], the Lorentz Transformations (LT) render covariant Maxwell's equations, and so predict that electromagnetic waves have the same speed in vacuum, c, in all inertial frames. In Einstein's seminal paper on SRT on 1905 [5] this prediction was elevated (though tacitly[1]) to the status of a postulate and used to derive the LT from first principles.

It was soon realised that Einstein's 'second postulate' concerning the constancy of the speed of light was not necessary in order to derive the LT. Following the pioneering papers of Ignatowski [6], Frank and Rohe [7] and Pars [8] the number of published axiomatic derivations of the LT, not invoking light-speed-constancy, is now vast. Work prior to 1968 is cited in Ref. [9], more recent work in Ref. [10] by the present author. The majority of these derivations, following that of Einstein in 1905, were 'kinematical' in the sense that no dynamical laws of physics, or the properties of particular physical systems, were considered. In contrast, in the original applications of the LT by Lorentz [2] and Poincaré [4] to the electromagnetic structure of the electron, kinematics and dynamics were inextricably intertwined. It was stressed by Pais [11] that one of the the most important conceptual advances contributed by Einstein to the development of SRT was the demonstration of the 'kinematical' and hence universal nature of the LT.

Advantages of axiomatic derivations of the LT are the simplicity and almost 'obvious' nature of the postulates used, and mathematical elegance. However a disadvantage, presciently pointed out in 1921 by Pauli [12] is that such derivations shed little light on the physical meaning of the equations:

> `From the group theoretical assumption it is only possible to determine`
> `the general form of the transformation formulae not of their physical`
> `content.`

Indeed the still controversial and anti-common-sense nature of some predictions of SRT are due, not to the equations of the theory *per se*, but rather how the equations are physically interpreted.

In the work of Einstein, and in particular in axiomatic derivations of later authors, as well as in text books, the LT are presented as transformations between an *event*:

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[1]The actual statement of Einstein's 'principle of the constancy of the speed of light' is [5]: 'Any ray of light moves in the "stationary" system of coordinates with a determined velocity c whether the ray is emitted by a stationary or by a moving body'. This is only a statement of source-speed-independence of the speed of light in a particular frame. In the subsequent derivation of the LT in Ref. [5] the equality of the speed of light in the 'stationary' system K and the 'moving' system k is asserted without further comment.

(x,y,z,t) observed in one frame S and the *same event*: (x',y',z',t') as observed in another frame S'. In order to specify such an event it is evident that spatial coordinate systems must be defined in the different frames. Also the existence of the epochs t, t' of the occurence of the event presuppose the existence of some clocks, at rest in the frames S, S' respectively, to measure them. The problem of the synchronisation of these clocks, in relative motion, must then be addressed. As pointed out in previous work by the present author [13, 14, 15, 16, 17], and as will be seen below, certain incorrect predictions of SRT that have survived, in the literature, text books and the popular understanding of SRT from 1905 until the present day, are due not to any errors in the LT *per se*, but rather incorrect handling of coordinate systems and clock synchronisation constants when they are used to transform space time *events* rather than space time *intervals*.

The present paper presents another axiomatic derivation of SRT where firstly the formulas for relativistic energy, E, and momentum, $\mathbf{p}$ are obtained, as well as transformation laws relating different kinematical configurations $(E,\mathbf{p})$ of the same physical object. In a second step the time dilation (TD) relation and the time LT for space and time intervals are derived. Thirdly, on introducing spatial coordinate systems, the LT for space-time events are derived. Lastly, relativistic transformations of *relative velocities*, as previously discussed in the literature, are derived and discussed. Unlike in previous work, the axioms employed make no reference to the special relativity principle, light speed, group properties, or any considerations of spatial geometry or of symmetry. They are:

(I) Work-to-energy conversion: $dE = dW = \mathbf{F} \cdot d\mathbf{s}$.

(II) Newton's Second Law of mechanics: $d\mathbf{p}/dt = \mathbf{F}$.

(III) Ansätze, proportional to its Newtonian mass, m, for the velocity dependence of the relativistic energy, E, and momentum, $\mathbf{p}$, of a physical object.

Here dE is the change of energy of a physical object due to the work dW done by its displacement $d\mathbf{s}$ under the action of an impressed force $\mathbf{F}$. The derivation proceeds by first obtaining expressions for E and $\mathbf{p}$ as a function of velocity and then considering the transformation $(E,\mathbf{p}) \rightarrow (E',\mathbf{p}')$ of kinematical configurations of the object in a single reference frame, due to the action of the force. The transformation law so obtained allows the definition of two kinematical invariants, from the first of which the TD relation and the LT for time intervals may be obtained. Because these transformtion laws contain only space or time *intervals* these equations unlike the LT for *events* do not depend on the definition of coordinate systems or of clock synchronisation constants. On introducing coordinate systems and synchronised clocks the LT for space-time events are then obtained and compared with the conventional event LT of SRT. Finally, predictions of space-time geometry (i.e. for observations of spatial postions at different epochs of the same object in different inertial frames in the same space-time experiment) are compared and contrasted with those of relativistic kinematics. In particular transformation formulas for relative velocities are derived and compared with the velocity transformation formulas of conventional SRT as derived by Einstein [5].

The organisation of the paper is as follows: In the following section explicit formulas for relativistic energy and momentum are derived by adapting a calculation of Feynman [18]

invoking Newton's Second Law. An initially unknown universal constant occuring, for dimensional reasons, in the initial ansätze is identified with the square of the speed of light in free space. In Section 3 the transformation laws of relativistic energy and momentum are derived from an invariant relation obtained in the previous section. In Section 4 the space-time LT for intervals or events are derived and interpreted to demonstrate the spurious nature of the 'length contraction' and 'relativity of simultaneity' effects of conventional SRT. In Section 5, space-time geometrical and kinematical velocity transformations are compared and contrasted. Group properties of different velocity transformations and their physical interpretation are discussed in an Appendix.

2   Relativistic energy and momentum

Relativistic energy and momentum as attributes of a physical object of Newtonian mass m, and velocity v in some inertial frame, are defined here according to the ansätze:

$$E(v) \equiv \kappa m \gamma(v), \tag{2.1}$$

$$\mathbf{p}(v) \equiv \frac{E(v)}{\kappa}\mathbf{v} = m\gamma(v)\mathbf{v}. \tag{2.2}$$

In these equations κ is a universal constant of dimensions $L^2 T^{-2}$, the physical significance of which will be elucidated in the following. The initially unknown, dimensionless, function $\gamma(v)$, ($v \equiv |\mathbf{v}|$) may be assumed, because it is multiplied in Eq. (2.1) by the universal constant κ, to satisfy the condition $\gamma(0) = 1$. Clearly, any other choice for the value of $\gamma(0)$ will only result in a different value of κ.

Following a proof due to Feynman [18], the function $\gamma(v)$ and hence the velocity dependence of E and $\mathbf{p}$ will now be determined from the solution of a first order differential equation obtained by combining the similar equations describing the conversion of work into energy:

$$dE = dW = \mathbf{F} \cdot d\mathbf{s} \tag{2.3}$$

and Newton's Second Law of mechanics, expressed in terms of momentum:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}. \tag{2.4}$$

The quantity $d\mathbf{s}$ is the infinitesmal displacement of the object during the time interval dt due to the action of an arbitary force $\mathbf{F}$. Since $\mathbf{v} \equiv d\mathbf{s}/dt$ (2.1)-(2.4) may be combined to give:

$$dE = \kappa m d\gamma = \mathbf{F} \cdot d\mathbf{s} = \mathbf{v} \cdot d\mathbf{p} = m\mathbf{v} \cdot d(\gamma \mathbf{v}). \tag{2.5}$$

Cancelling throughout the common factor m and multiplying through by γ, (2.5) gives:

$$\kappa \gamma d\gamma = \frac{\kappa}{2}d(\gamma^2) = \gamma \mathbf{v} \cdot d(\gamma \mathbf{v}) = \frac{1}{2}d(\gamma^2 v^2) \tag{2.6}$$

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so that $\gamma(v)$ is the solution of the differential equation:

$$\kappa d(\gamma^2) = d(\gamma^2 v^2). \tag{2.7}$$

Direct integration of (2.7) with the boundary condition $\gamma(0) = 1$ gives:

$$\kappa[\gamma(v)^2 - 1] = \gamma(v)^2 v^2. \tag{2.8}$$

Solving this equation for $\gamma(v)$ it is found that:

$$\gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{\kappa}}}. \tag{2.9}$$

The above calculation is identical to that given by Feynman [18] except that Feynman assumed initially that $\kappa = c^2$ where c is the speed of light in vacuum. In the present work this identification [10] will be seen to follow by assuming that light actually consists of particles —photons.

Since the relation (2.9) is physical only if $v < \sqrt{\kappa} = v_{\mathrm{MAX}}$ the physical significance of the constant κ is clear: $\kappa = v_{\mathrm{MAX}}^2$ *where v_{MAX} is the maximum possible velocity of a ponderable object in an inertial frame allowed by the application of forces that obey Newton's Second Law.*

To identify the limiting velocity v_{MAX} with the speed of light in free space (2.2) is written as:

$$p = \sqrt{\kappa} m \gamma(v) \beta(v) \tag{2.10}$$

where $\beta(v) \equiv v/\sqrt{\kappa}$. Then use of the identity: $\gamma^2 - \gamma^2 \beta^2 \equiv 1$ together with (2.1) and (2.10) gives

$$E^2 - \kappa p^2 = \kappa^2 m^2 (\gamma^2 - \gamma^2 \beta^2) = \kappa^2 m^2. \tag{2.11}$$

For any object which is light [2] in the sense that $E, \sqrt{\kappa} p \gg \kappa m$, (2.1), (2.2) and (2.11) give the relation

$$v = \frac{\kappa p}{E} \simeq \frac{\kappa p}{\sqrt{\kappa} p} = \sqrt{\kappa} = v_{\mathrm{MAX}}. \tag{2.12}$$

If light is therefore identified as the manifestation of the propagation of massless (or almost massless) particles in free space it follows that $c \simeq v_{\mathrm{MAX}} = \sqrt{\kappa}$. The second of Einstein's postulates of special relativity, when (see below) it is correctly physically interpreted, is therefore *derived* in the present approach. An experimentally verified corollory is that *all* particles satisfying the conditions $E, \sqrt{\kappa} p \gg \kappa m$ will move isotropically, in the inertial frame in which they are produced, at the same speed as light in free space.

3 Transformation laws of relativistic energy and momentum

It follows from Eq. (2.11) that different kinematical configurations, $(E,\mathbf{p})$, $(E',\mathbf{p}')$ of an object of Newtonian mass m are related by the equation:

$$E^2 - \kappa p^2 = (E')^2 - \kappa(p')^2 = \kappa^2 m^2. \tag{3.1}$$
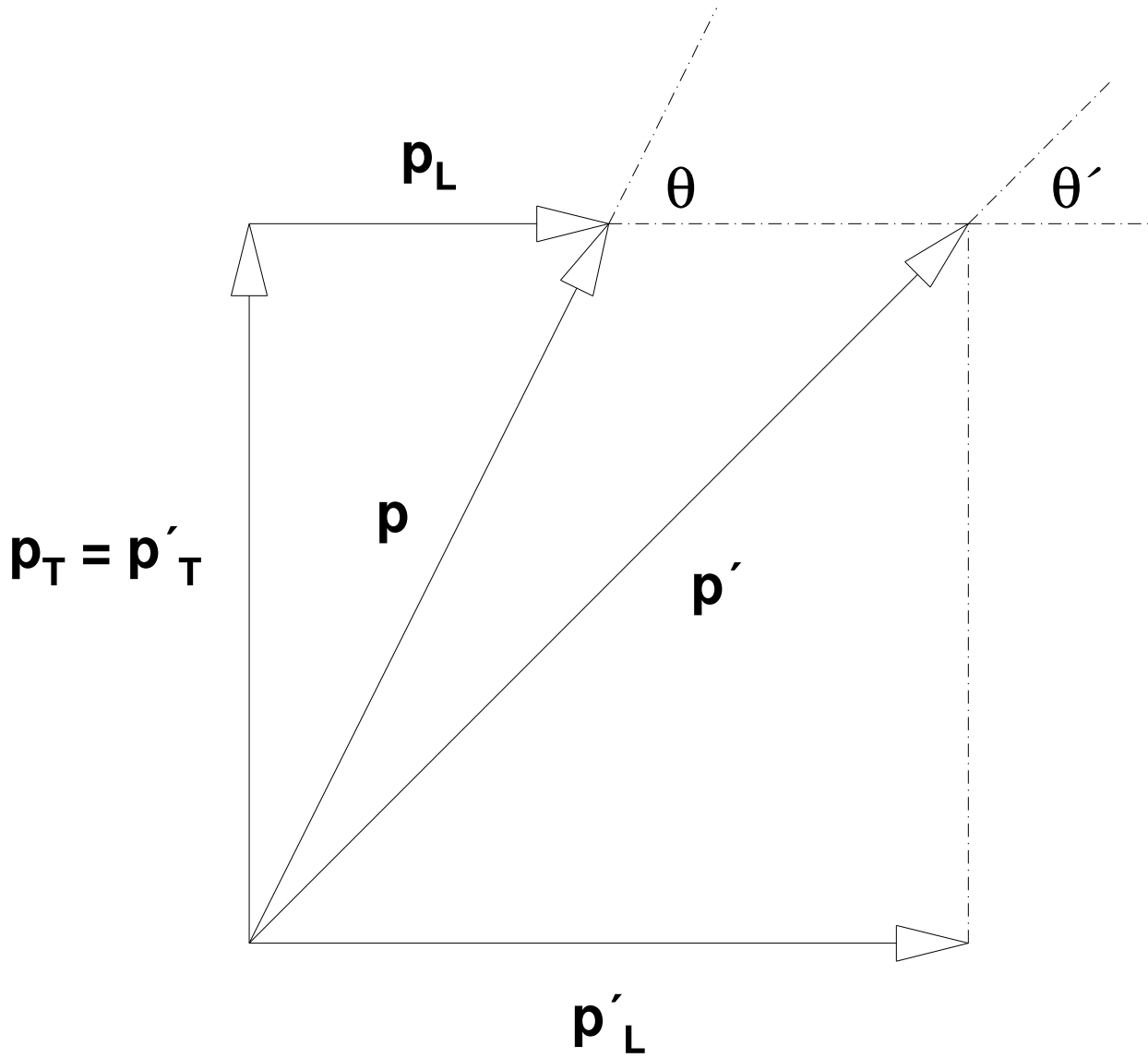
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[2]No pun intended

Figure 1: *Geometrical definitions for the transformation of relativistic energy-momentum configurations:* $(E, \mathbf{p}) \rightarrow (E', \mathbf{p}')$. *Momenta and angles are shown in the plane spanned br the relativistic momentum vectors* $\mathbf{p}$ *and* $\mathbf{p}'$.

It is germane to consider generation of the configuration $(E',\mathbf{p}')$ from $(E,\mathbf{p})$ by a two parameter boost operation. Since the momentum vectors $\mathbf{p}$ and $\mathbf{p}'$ define a plane in the inertial frame of observation it is convenient, as shown in Fig. 1, to decompose these vectors into components parallel to (longitudinal, L, component) and transverse to (T component) the vector $\mathbf{p}' - \mathbf{p}$:

$$\mathbf{p} = \mathbf{p}_\mathrm{L} + \mathbf{p}_\mathrm{T}, \quad \mathbf{p}' = \mathbf{p}'_\mathrm{L} + \mathbf{p}'_\mathrm{T}, \tag{3.2}$$

$$p_\mathrm{L} = p\cos\theta, \quad p'_\mathrm{L} = p'\cos\theta', \quad p_\mathrm{T} = p\sin\theta = p'_\mathrm{T} = p'\sin\theta'. \tag{3.3}$$

It therefore follows from (3.1)-(3.3) that there are two independent frame-invariant relations [19] involving, respectively, the longitudinal and transverse components of relativistic momentum:

$$E^2 - \kappa p_\mathrm{L}^2 = (E')^2 - \kappa(p'_\mathrm{L})^2, \tag{3.4}$$
$$p_\mathrm{T}^2 = (p'_\mathrm{T})^2. \tag{3.5}$$

Linear transformation equations between $(E,\mathbf{p}_\mathrm{L})$ and $(E',\mathbf{p}'_\mathrm{L})$ may, in general, be written as:

$$E' = c_1 E + c_2\sqrt{\kappa}p_\mathrm{L}, \tag{3.6}$$
$$p'_\mathrm{L} = c_3 p_\mathrm{L} + \frac{c_4 E}{\sqrt{\kappa}} \tag{3.7}$$

where c_1, c_2, c_3 and c_4 are dimensionless coefficients. Setting $E = m\kappa$ and $p_\mathrm{L} = 0$ in (3.6) gives, on comparing with the relations $E' = \kappa m\gamma'$, $p' = \sqrt{\kappa}m\gamma'\beta'$: $c_1 = \gamma'$ and $c_4 = \gamma'\beta'$. This suggests writing the transformation equations as:

$$E' = \gamma''(E + a\sqrt{\kappa}p_\mathrm{L}), \tag{3.8}$$
$$p'_\mathrm{L} = \gamma''(bp_\mathrm{L} + \frac{\beta'' E}{\sqrt{\kappa}}) \tag{3.9}$$

where the two parameters specifying the boost are θ and β'', and the dimensionless coefficients a and b are determined by consistency of (3.8) and (3.9) with (3.4). Substituting (3.8) and (3.9) in (3.4) it is found that γ'' and β'' satisfy the identity: $(\gamma'')^2 - (\gamma''\beta'')^2 \equiv 1$, and that $a = \beta''$ and $b = 1$, so that the complete transformation equations of relativistic energy and momentum are:

$$E' = \gamma''(E + \beta''\sqrt{\kappa}p_\mathrm{L}), \tag{3.10}$$
$$p'_\mathrm{L} = \gamma''(p_\mathrm{L} + \frac{\beta'' E}{\sqrt{\kappa}}), \tag{3.11}$$
$$p'_\mathrm{T} = p_\mathrm{T}. \tag{3.12}$$

On setting $\sqrt{\kappa} = c$, the well-known transformation equations of relativistic energy and momentum are recovered.

Noting that $\beta = \sqrt{\kappa}p/E$, $\beta' = \sqrt{\kappa}p'/E'$ and $\beta'' = \sqrt{\kappa}p''/E''$, multiplying (3.11) by $\sqrt{\kappa}$ and dividing it by (3.10) gives:

$$\beta'\cos\theta' = \frac{\beta\cos\theta + \beta''}{1 + \beta''\beta\cos\theta}, \tag{3.13}$$

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while dividing (3.12) by (3.10) gives

$$\beta' \sin \theta' = \frac{\beta \sin \theta}{\gamma''(1 + \beta'' \beta \cos \theta)}. \tag{3.14}$$

Solving (3.13) for β''. gives the boost parameter β'' in terms of the parameters (β,θ) and (β',θ') specifying the configurations:$(E,\mathbf{p})$ and $(E',\mathbf{p}')$:

$$\beta'' = \frac{\beta' \cos \theta' - \beta \cos \theta}{1 - \beta \beta' \cos \theta \cos \theta'}. \tag{3.15}$$

The ratio of (3.14) to (3.13) gives the angle θ' in terms of the boost parameters (θ, β'') and the parameters (β,θ) of the original configuration $(E,\mathbf{p})$:

$$\tan \theta' = \frac{\beta \sin \theta}{\gamma''(\beta \cos \theta + \beta'')}. \tag{3.16}$$

It is important to contrast the physical meaning of the relations (3.10)-(3.16), just derived, from the interpretation given to them in conventional special relativity theory. The configurations $(E,\mathbf{p})$ and $(E',\mathbf{p}')$ are different ones of the same object as viewed in the same reference frame. The physical meaning of the boost (θ,β'') is similar to that of an active rotation of the position vector of an object in a fixed reference frame. An invariant of the latter transformation is the length of the position vector; the invariant of the transformation between $(E,\mathbf{p})$ and $(E',\mathbf{p}')$ is that defined by Eq. (3.1) above. In contrast (3.10) to (3.12) have previously been interpreted as giving the energy and momentum $(E',\mathbf{p}')$ of an object as observed in a frame with boost $(\theta,-\beta'')$ relative to the frame with configuration $(E,\mathbf{p})$. In a similar manner, (3.15) has been erroneously interpreted as giving the relative velocity, in the frame where the object has the configuration $(E',\mathbf{p}')$, of the frame where the configuration is $(E,\mathbf{p})$. This problem of interpretation will be rediscussed below after consideration of the space-time transformation equations.

4   Derivation and interpretation of the space-time Lorentz transformation equations

The second member of Eq. (2.11) may be written as:

$$[\kappa m \gamma(v)]^2 - \kappa[m \gamma(v)v]^2 = (\kappa m)^2. \tag{4.1}$$

Dividing through by a factor κm^2 and using the definition of the velocity v: $v \equiv ds/dt$ where ds is a spatial displacement of the object, gives:

$$\kappa \gamma(v)^2 - \left[\gamma(v)\frac{ds}{dt}\right]^2 = \kappa \tag{4.2}$$

or, on re-arrangement:

$$\kappa(dt)^2 - (ds)^2 = \kappa(d\tau)^2 \tag{4.3}$$

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where:
$$d\tau \equiv \frac{dt}{\gamma(v)}. \tag{4.4}$$

Inspection of (4.3) shows that $dt = d\tau$ when $ds = 0$, so that $d\tau$ is a time interval in the rest frame of the object. Transposition of (4.4) leads to the time dilation (TD) relation:

$$dt = \gamma(v)d\tau \tag{4.5}$$

relating a proper time interval $d\tau$ (a time interval registered by a clock at rest relative to the object) to the time interval dt recorded by a clock, relative to which, the object moves at speed v. It is important to note that the TD relation has no spatial dependence so that on integrating (4.5) to give the relation between clock settings: $t = \gamma(v)\tau$, an arbitary pair of clocks 1, 2 with a common velocity (and so at rest, at any postion, in the same frame) that are synchronised so that at a given instant $\tau_1 = \tau_2$ are also observed to be so by a stationary observer:

$$t_1 = \gamma(v)\tau_1 = \gamma(v)\tau_2 = t_2$$

—there is no 'relativity of simultaneity' effect.

Space and time intervals along the worldline of a moving clock that obeys the TD relation (4.5) satisfy the relation:

$$ds = vdt = \sqrt{\kappa}\beta dt. \tag{4.6}$$

Combining this equation with (4.4) yields the Lorentz transformation equation for time intervals:

$$d\tau = \frac{dt}{\gamma(v)} = \frac{\gamma(v)dt}{\gamma(v)^2} = \gamma(v)(dt - \beta^2 dt) = \gamma(v)\left(dt - \frac{\beta ds}{\sqrt{\kappa}}\right). \tag{4.7}$$

Denoting, as is conventional, by S the inertial frame in which the clock has speed v, and S' (with interval of proper time $dt' = d\tau$ and displacement ds') that in which it is at rest, then the Lorentz transformation for spatial intervals corresponding to (4.7) is:

$$ds' = \gamma(v)(ds - vdt) = 0 \tag{4.8}$$

since the displacement, ds', of any object at rest in S' vanishes. Note that the factor $\gamma(v)$ on the on the right side of (4.8) can be replaced by any finite constant without affecting any physical consequence of the transformations (4.7) and (4.8). In fact, these equations are physically equivalent, respectively, to:

$$d\tau = \frac{dt}{\gamma(v)}, \tag{4.9}$$
$$ds' = 0, \qquad ds = vdt. \tag{4.10}$$

The differential worldline equation (4.10) is simply a description of uniform motion in the frame S, and so is the same in special and Galilean relativity. Then the only modification of space-time geometry in passing from Galilean to special relativity is the replacement of an interval of Newtonian universal time: $dT = dt = d\tau$ by the transposed TD relation (4.9).

In order to relate the transformation equations for time and space *intervals* (4.7) and (4.8) to those relating observation of a space time *event* in the frame S to the same event as observed in the frame S', spatial coordinate systems must be introduced in these frames to specify the positions of events. Choosing Cartesian coordinate systems with arbitary origins but parallel axes, enables space-time coordinates of an event $(x,y,z;t)$ in S and $(x',y',z';t')$ in S' to be specified. With the direction of the x and x' axes parallel to **v**, the worldline of an object at rest in S' is written in the frame S as:

$$
\begin{aligned}
x(t) &= x(t_0) + v(t - t_0), & (4.11) \\
y(t) &= y(t_0), & (4.12) \\
z(t) &= z(t_0) & (4.13)
\end{aligned}
$$

and in the frame S' as:

$$
\begin{aligned}
x'(t) &= x'(t_0), & (4.14) \\
y'(t) &= y'(t_0), & (4.15) \\
z'(t) &= z'(t_0). & (4.16)
\end{aligned}
$$

In virtue of the linearity of the interval transformation equations, (4.7) and (4.8) may be written in terms of the coordinates introduced in (4.11)-(4.16) as:

$$
\Delta\tau \equiv \tau(t) - \tau(t_0) = t'(t) - t'(t_0) = \gamma(v)\left[t - t_0 - \frac{\beta}{\sqrt{\kappa}}[x(t) - x(t_0)]\right], \quad (4.17)
$$

$$
\Delta s' \equiv x'(t) - x'(t_0) = \gamma(v)[x(t) - x(t_0) - v(t - t_0)] = 0 \quad (4.18)
$$

and (4.9) and (4.10) as:

$$
\Delta\tau \equiv \tau(t) - \tau(t_0) = t'(t) - t'(t_0) = \frac{t - t_0}{\gamma(v)}, \qquad \Delta t \equiv t - t_0, \quad (4.19)
$$

$$
\Delta s' \equiv x'(t) - x'(t_0) = 0, \qquad \Delta s = x(t) - x(t_0) = v(t - t_0). \quad (4.20)
$$

The conventional space-time Lorentz transformation equations for space-time events, as derived by Einstein [5], to be found in text books on special relativity, correspond to a particular choice of coordinate origins and clock synchronisation constants in (4.17) and (4.18):

$$
x'(t_0) = x(t_0) = 0, \qquad t_0 = \tau(t_0) = 0
$$

to yield

$$
\begin{aligned}
t'(t) = \tau(t) &= \gamma(v)\left[t - \frac{\beta}{\sqrt{\kappa}}[x(t)]\right], & (4.21) \\
x'(t) &= \gamma(v)[x(t) - vt] = 0 & (4.22)
\end{aligned}
$$

which, on ignoring the temporal dependence of the coordinates and the proper time as well as the '= 0' in the last member of (4.22) gives, on setting $\sqrt{\kappa} = c$, the conventional Lorentz transformations:

$$
\begin{aligned}
t' = \tau &= \gamma(v)\left[t - \frac{\beta}{c}x\right], & (4.23) \\
x' &= \gamma(v)[x - vt]. & (4.24)
\end{aligned}
$$

The spurious 'length contraction' and 'relativity of simultaneity' effects arise essentially from the missing conjunction '$= 0$' on the right side of Eq. (4.24). Since x' specifies the position of an object in a frame in which it is rest, it must be independent of time. Since the right side of (4.24) vanishes for some value of t (when $t = x/v$), whatever the value of x, it must therefore, because the left side of (4.24) is independent of t, vanish for all values of t. *That is* (4.24)*is only valid when both* $x' = 0$ *and* $x = vt$.

The transformations (4.21) and (4.22) show how an event $(x' = 0, t' = 0)$ in the frame S' is observed as $(x = 0, t = 0)$ in the frame S. This means that clocks at rest placed at the origins of S and S' are synchronised so that $t = t' = 0$, when the origins are aligned in the x, x' direction. With the same choice of coordinate origins in the frames S and S' and the same synchronisation condition, but placing an object (on the worldline of which the transformed event is to be situated) instead at $x' = x'(0) \neq 0$ corresponds to the choice of initialisation constants in (4.17) and (4.18)[3]:

$$x'(t_0) = x(t_0) \neq 0, \qquad t_0 = \tau(t_0) = 0$$

to give the transformation equations:

$$t'(t) = \tau(t) \quad = \quad \gamma(v)\left[t - \frac{\beta}{\sqrt{\kappa}}[x(t) - x(0)]\right], \tag{4.25}$$

$$x'(t) - x(0) \quad = \quad \gamma(v)[x(t) - x(0) - vt] = 0 \tag{4.26}$$

or, equivalently:

$$t'(t) = \tau(t) \quad = \quad \frac{t}{\gamma(v)}, \tag{4.27}$$

$$x'(t) - x(0) = 0, \qquad x(t) - x(0) = vt. \tag{4.28}$$

The distance, in S', between an object at the origin of S' and one at $x' = x(0)$ is $x(0)$. As the objects are at rest in S' this separation, $\Delta x'$, is time-independent. Since the world line of an object at the origin of S' is, according to (4.22), $x(t) = vt$, it follows from (4.28) that, for all values of t:

$$\Delta x \equiv x[t, x' = x(0)] - x[t, x' = 0] = x(0) = \Delta x'. \tag{4.29}$$

There is therefore no 'length contraction' effect. See Refs. [13, 14, 15, 16, 17] for an explanation of how the spurious and correlated 'length contraction' and 'relativity of simultaneity' effects arise from manipulation of the conventional Lorentz transformation equations (4.23) and (4.24), under the erroneous assumption that $x' \neq 0$ in (4.24), so that these equations are erroneously assumed to be valid for all values (x,t) and (x',t').

5 Relativistic space-time geometry is distinct from relativistic kinematics

Consider observation of a point on the worldline of a moving object in two inertial

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[3]Note that the freedom to choose independently coordinate origins in the frames S and S' without changing any physical predictions (translational invariance) always allows the choice $x'(0) = x(0)$ for the initial conditions of the problem.
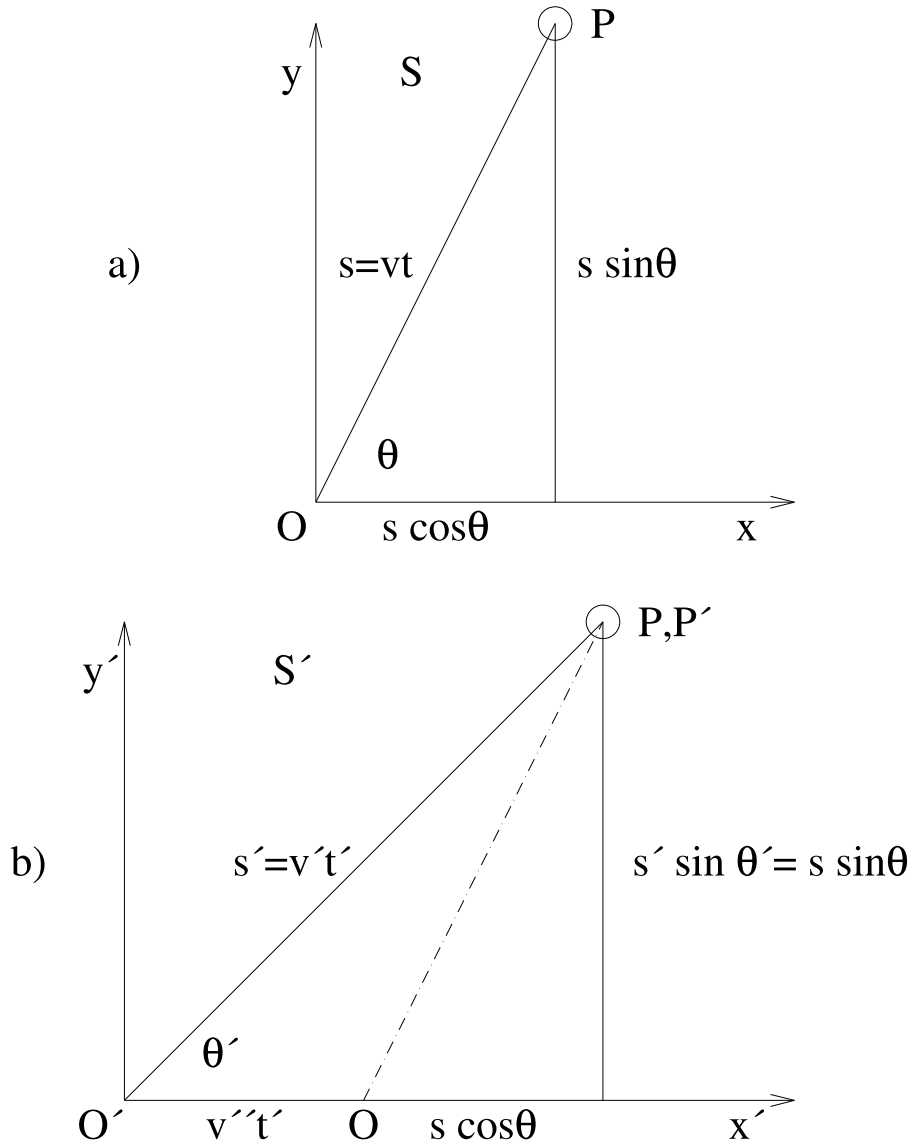
Figure 2: *Space-time geometry of the world-lines of a uniformly-moving object in different inertial frames: a) frame S, world-line segment OP, speed v; b) frame S',world-line segment O'P', speed v'. The frame S moves with speed v'' in S' along the common x,x' axis. See text for discussion.*

frames. Contrary to the usual convention as employed in the previous section, and in order to facilitate direct comparison with the kinematical transformations considered in Section 3, the inertial frame S is assumed to move uniformly with speed v'' along the x'-axis of the frame S'. As shown in Fig. 2, the Cartesian coordinate axes x,x' and y,y' are assumed to be parallel. In the following, without any loss of generality, only motion in the $x-y$ plane is considered. At time $t = t' = 0$ the origins O and O' of the coordinate systems are coincident. Starting at this instant, an object moves in a straight line at speed v from O to P in S [Fig. 2a)] and at speed v' from O' to P' in S'.[Fig. 2b)].

The postulates to be used in the following space-time calculation in order to relate velocities and angles observed angles in one frame to those observed in another are:

(i) The definition of uniform velocity:

$$\text{velocity} \equiv \mathbf{v} = \frac{\text{displacement of object}}{\text{elapsed time}} \equiv \frac{\mathbf{s}}{t}. \tag{5.1}$$

(ii) The TD relation (4.5) for elapsed time intervals in the frames S and S' defined above:

$$t' = \gamma'' t \equiv \gamma(v'')t. \tag{5.2}$$

Considering the x, x'-components of the displacement, inspection of Fig. 2b gives:

$$s' \cos \theta' = v't' \cos \theta' = v''t' + s \cos \theta = v''t' + vt \cos \theta. \tag{5.3}$$

Using (5.2) and re-arranging gives the transformation formula for x, x'-components of the velocities $\mathbf{v}$ and $\mathbf{v}'$:

$$v_x = v \cos \theta = \gamma''(v' \cos \theta' - v'') = \gamma''(v'_{x'} - v''). \tag{5.4}$$

For the $y - y'$-components of the displacement:

$$s' \sin \theta' = v't' \sin \theta' = s \sin \theta = vt \sin \theta \tag{5.5}$$

so that application of (5.2) gives:

$$v_y = v \sin \theta = \gamma'' v' \sin \theta' = \gamma'' v'_{y'}. \tag{5.6}$$

Denoting the relative velocity of the moving object and O as $\mathbf{u}$ in S and $\mathbf{u}'$ in S', so that

$$\mathbf{u} = \mathbf{v} = \hat{i}v_x + \hat{j}v_y, \tag{5.7}$$
$$\mathbf{u}' = \hat{i}(v'_{x'} - v'') + \hat{j}v'_{y'} \tag{5.8}$$

where $\hat{i},\hat{j}$ are unit vectors parallel to the $(x, x'),(y, y')$ axes, (5.4) and (5.6) may be combined to give a simple vector equation for the transformation of relative velocities:

$$\mathbf{u} = \gamma'' \mathbf{u}'. \tag{5.9}$$

Relations derived by combining Eqs. (5.4) and (5.6) are now compared with similar ones for transformations of velocity and angles as derived in Section 2 above using relativistic kinematics. The former set of relations are denoted by 'RSTG', for 'relativistic space-time geometry' the latter by 'RKIN' for 'relativistic kinematics'. Transformed quantities in RSTG relations are distinguished by a tilde accent, those in RKIN relations with a circumflex accent. For ease of comparison, all velocities are scaled by the factor $1/\sqrt{\kappa}$

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- velocity transformation formulas

$$\tilde{\beta}'_{x'} = \frac{\beta_x}{\gamma''} + \beta'', \qquad\qquad \tilde{\beta}_x = \gamma''(\beta'_{x'} - \beta''), \qquad \text{(RSTG)} \qquad (5.10)$$

$$\hat{\beta}'_{x'} = \frac{\beta_x + \beta''}{1 + \beta''\beta_x}, \qquad\qquad \hat{\beta}_x = \frac{\beta'_{x'} - \beta''}{1 - \beta''\beta'_{x'}}, \qquad \text{(RKIN)} \qquad (5.11)$$

$$\tilde{\beta}'_{y'} = \frac{\beta_y}{\gamma''}, \qquad\qquad \tilde{\beta}_y = \gamma''\beta'_{y'}, \qquad \text{(RSTG)} \qquad (5.12)$$

$$\hat{\beta}'_{y'} = \frac{\beta_y}{\gamma''(1 + \beta''\beta_x)}, \qquad\qquad \hat{\beta}_y = \frac{\beta'_{y'}}{\gamma''(1 - \beta''\beta'_{x'})}. \qquad \text{(RKIN)} \qquad (5.13)$$

- angle transformation formulas

$$\tan\tilde{\theta}' = \frac{\beta\sin\theta}{\beta\cos\theta + \gamma''\beta''}, \qquad\qquad \tan\tilde{\theta} = \frac{\beta'\sin\theta'}{\beta'\cos\theta - \beta''}, \qquad \text{(RSTG)} \quad (5.14)$$

$$\tan\hat{\theta}' = \frac{\beta\sin\theta}{\gamma''(\beta\cos\theta + \beta'')}, \qquad\qquad \tan\hat{\theta} = \frac{\beta'\sin\theta'}{\gamma''(\beta'\cos\theta' - \beta'')}. \qquad \text{(RKIN)} \quad (5.15)$$

In these formulas the unaccented velocity symbols correspond to initial conditions in the space-time experiments described, i.e. to velocities of objects created or placed in certain reference frames (the frames S or S' in Eqs. (5.10)-(5.15)). All such velocities therefore respect the condition: $-1 \le \beta \le 1$ imposed by Eq. (2.9) above. The same condition is seen to be respected by the transformed velocities $\hat{\beta}$ in the RKIN formulas. Since, however, the velocities $\tilde{\beta}$ in the RSTG formulas are, by definition *relative velocities of different objects in the same reference frame* they are not subject to the constraint imposed by Eq. (2.9). It fact. it is found that:

$$-\sqrt{2} \le \tilde{\beta}'_{x'} \le \sqrt{2}, \qquad\qquad -\infty \le \tilde{\beta}_x \le \infty, \qquad\qquad (5.16)$$
$$0 \le \tilde{\beta}'_{y'} \le 1, \qquad\qquad 0 \le \tilde{\beta}_y \le \infty. \qquad\qquad (5.17)$$

The limits on $\tilde{\beta}'_{x'}$ are found by differentiating the first formula in (5.10) w.r.t. β'', to find the maximum value of the function of β'' on the right side of this equation. It is found to occur when $\beta_x = 1$ and $\beta'' = 1/\sqrt{2}$. Note that with the coordinate axes shown in Fig. 2, β_y and $\beta'_{y'}$ are positive quantities.

Neglecting terms of $O(\beta^2)$ and higher in (5.10)-(5.15) (that is in Galilean relativity) gives identical space-time-geometrical and kinematical transformation formulas. Unlike the RKIN formulas the RSTG formulas and their inverses are not form-invariant. The RSTG formula (5.10) does not respect the 'Reciprocity Principle' [9] that holds in Galilean relativity. Setting $\beta'_{x'} = 0$ in (5.11) gives gives $\hat{\beta}_x = -\beta''$ i.e. the origin of S' moves with velocity $-\beta''$ in the x-direction when (see Fig. 2) the origin of S moves with velocity β'' in the x'-direction. Setting $\beta'_{x'} = 0$ in the RSTG formula (5.10) gives instead $\tilde{\beta}_x = -\gamma''\beta''$. This violates the Reciprocity Principle that states that: 'If the velocity of B as observed by A is **u** then the velocity of A as observed by B is $-\mathbf{u}$'

As demonstrated in the Appendix, the RSTG formulas, unlike the RKIN ones, do not constitute a group.

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The RSTG transformation formulas of Eq. (5.10) for the case $\theta = \theta' = 0$ have previously been derived in the context of the analysis of a circular Sagnac interferometer by Post [20] and Klauber [21]. Setting $\theta = \theta' = 0$, $\beta'_{x'} = 1$ in the RKIN formula (5.11) gives $\beta_x = 1$, so predicting in accordance with the tacit hypothesis made in Einstein's original 1905 paper on special relativity [5] that: 'the speed of light is the same in all inertial frames'. The incompatiblity of this hypothesis with with the interference effect observed by Sagnac [22], that is correctly predicted by the RSTG formulas in (5.10) (or by their $O(\beta)$ approximations) was already pointed out in 1937 by Dufour and Prunier [23], as well as, more recently, by Selleri [24], Klauber [21] and the present author [25].

According to the argument given at the end of Section 3 of the present paper, all particles with rest energy much smaller than their actual energy have a fixed speed c in any inertial frame —apparently therefore confirming the correctness of Einstein's second postulate of special relativity. How now is the apparent paradox that the latter is inconsistent with the results of Sagnac's experiment to be resolved? The answer lies in careful consideration of the exact operational definitions of the various symbols appearing in the transformation equations (5.10)-(5.15). Since the unaccented velocities are all those of ponderable or massless objects either created or placed in one of the inertial frames S or S', they are subject to the restriction of Eq, (2.9): $-1 \leq \beta \leq 1$. However, in the RSTG equations the velocities $\tilde{\beta}$ and $\tilde{\beta}'$ are, in all cases, *relative velocities of two distinct objects in the same inertial frame* as compared to the unaccented velocities which are instead 'frame velocities', i.e. *the velocity of a single object in an inertial frame as fixed by the initial conditions of the space-time experiment under consideration.* For example, in the second equation in (5.10), $\beta'_{x'} - \beta''$ is the x' component of the relative velocity of the origin O of the frame S. and the object considered in Fig. 2, in the frame S'. The same relative velocity, as observed in the frame S is $\tilde{\beta}_x$. The RSTG equations do relate observations of relative velocites of the same two objects in different frames, in the same experiment. In contrast, as is clear from the considerations of of Section 3 above the RKIN equations instead relate different kinematical configurations $(E,\mathbf{p})$ and $(E',\mathbf{p}')$, in the same frame, of a single object, via the boost parameters (θ,β''). The application of Newton's second law in the derivation of Eq. (3.1) shows that the two configurations are both specified in the same reference frame: the frame in which this law is applied in Eq. (2.5). The situation is analogous to the active rotation of a spatial vector $\mathbf{r}$ into another vector $\mathbf{r}'$ of the same length, by rotation through an angle θ. The invariant of the transformation is:

$$\mathbf{r} \cdot \mathbf{r} = \mathbf{r}' \cdot \mathbf{r}'$$

to be compared with the invariant relation (3.1) for relativistic energy and momentum.

The experimental correctness of the RSTG equations and the inapplicabilty of the RKIN ones, for the case of photon propagation in the vicinity of the Earth is vouchsafed not only by Sagnac interferometers but also by GPS operation [26] and time transfer by microwave signals via a satellite in low-Earth orbit [27, 28, 29]. It is a prediction of the Schwartzschild metric equation of General Relativity [30, 31] that the ECI (Earth Centered Inertial) frame[4] is a preferred one for the propagation of light at speeds close to c in the vicinity of the Earth. The velocity of light signals in the frame of a receiver fixed on the surface of the Earth, and therefore rotating in the ECI frame, is given by the RSTG

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[4]This is a frame co-moving with the centroid of the Earth with axes pointing in fixed directions relative to the Celestial Sphere.

not the RKIN velocity transformations. Indeed, in GPS operation the Galilean (order v/c) approximation to the RSTG formulas is used [26]. The possibility to further test the RSTG predictions in long-baseline neutrino beams has been discussed in a recently published paper [32].

To give a concrete example of the very different predictions for space-time effects of the RSTG and RKIN velocity transformations consider the case of two protons colliding, head on, in the LHC collider at CERN. Suppose that they are each, initially, at a distance of 5m from their collision point. Each one is assumed to have an energy, as during the experimental program of 2012, of 4 TeV, corresponding to a velocity in the laboratory such that $v/c = 1 - 5.5 \times 10^{-8}$. The *relative* velocity of the two protons in the laboratory fame S is then $2c(1 - 5.5 \times 10^{-8})$. According to Eq. (5.10) the velocity of the left-moving proton in the rest frame, S', of the right-moving one is:

$$\gamma(v)2c(1 - 5.5 \times 10^{-8}) \simeq 8528c$$

and since length intervals are invariant, the initial spatial separation of the protons, in both S and S', is 10m. In the laboratory frame, the time interval $\Delta t = 5m/[c(1 - 5.5 \times 10^{-8})] \simeq$ 16ns elapses until the protons collide. The corresponding time interval in the frame S', according to the TD relation is $\Delta t' = \Delta t/\gamma(v) = 3.75$ps, in agreement with the relative velocity and spatial separation in this frame just quoted. Using instead the RKIN formula (5.11) to calculate the relative velocity of the two protons in the frame S' gives $c(1 - 1.35 \times 10^{-15})$. According to the TD relation this corresponds to an initial separation $\simeq c\Delta t' = 1.13$mm of the colliding protons in the frame S', instead of 10m. Finally it may be noted that, if a conceptual 'ruler' of length L' in the frame S', is used to measure the initial separation of the protons in this frame, then according to the length contraction effect of conventional SRT, the initial separation of the protons in the frame S which is, by hypothesis, 10m, must be equal to $L'/\gamma(v)$ so that $L' = 4264 \times 10m =$ 42.6km, to be compared with $L' = 1.13$mm as required by TD and the RKIN velocity transformation formula! So consistent predictions for observations in the frames S and S' are obtained by application of the RSTG velocity transformations, whereas contradictory predictions follow from application of the RKIN transformation formulas, time dilation and the conventional length contraction effect.
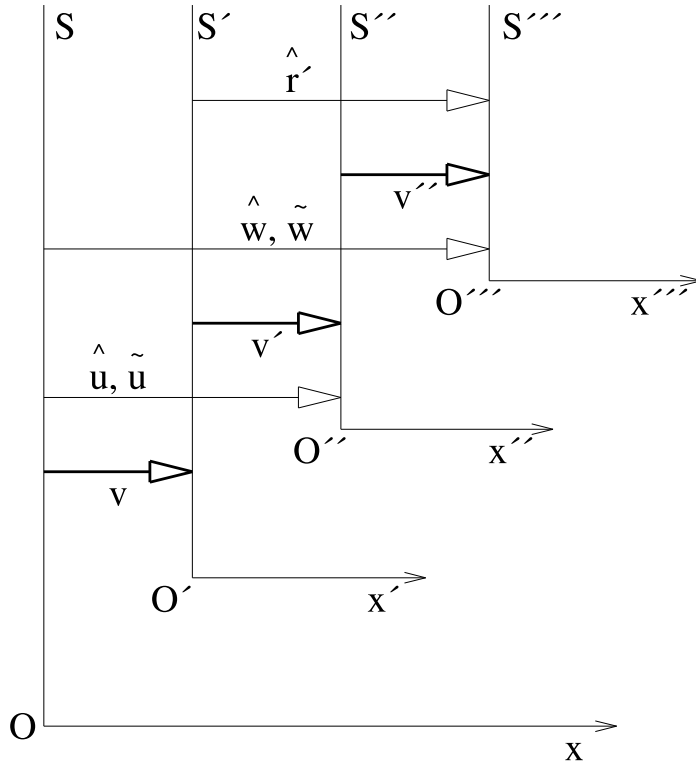
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Figure 3: *Inertial frames S', S'', S''' move parallel to the x axis of the frame S. The velocity of S' relative to S in S, v, of S'' relative to S' in S', v', and of S''' relative to S'' in S'', v'' (indicated by bold arrows) are fixed initial conditions. Application of the RSTG transformation formulas gives velocities $\tilde{u}$, $\tilde{w}$ in S, while application of the RKIN formulas gives velocities $\hat{u}$, $\hat{w}$ in S and $\hat{r}'$ in S'. See text for discussion.*

In order to discuss the group properties and physical interpretation of the RSTG and RKIN velocity transformation formulas in Eqs. (5.10) and (5.11) it is convenient to set $\theta = \theta' = 0$ (velocities parallel to the boost direction) and consider a nested series of inertial frames: S, S', S'', S''' as shown in Fig. 3. The relative velocities v, v' and, v'' of SS'', $S'S''$ and S'', S''', respectively, are fixed initial conditions and the velocities $\tilde{u}$ and $\hat{u}$ of S'' relative to S in the frame S or $\tilde{w}$ and $\hat{w}$ of S''' relative to S, are calculated according to the first formula in (5.10) (for $\tilde{u}$ and $\tilde{w}$) or the first formula in (5.11) (for $\hat{u}$ and $\hat{w}$). This gives, in an evident notation:

$$\hat{\beta}_u = \frac{\beta_v + \beta_{v'}}{1 + \beta_v \beta_{v'}}, \qquad \hat{\beta}_w = \frac{\hat{\beta}_u + \beta_{v''}}{1 + \hat{\beta}_u \beta_{v''}}. \qquad \text{(RKIN)} \tag{A.1}$$

Substituting $\hat{\beta}_u$ from the first of these equations in the second one it is found that

$$\hat{\beta}_w = \frac{\beta_v + \hat{\beta}_{r'}}{1 + \beta_v \hat{\beta}_{r'}} \tag{A.2}$$

where

$$\hat{\beta}_{r'} = \frac{\beta_{v'} + \beta_{v''}}{1 + \beta_{v'} \beta_{v''}}. \tag{A.3}$$

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Comparing (A.1)-(A.3) shows that successive applications of the transformation relations (substituting $\hat{\beta}_u$ in the formula for $\hat{\beta}_w$ in (A.1)) gives the form-invariant equation for $\hat{\beta}_w$, again in terms of velocities in S (β_v) and S' ($\hat{\beta}_{r'}$). The RKIN transformations are therefore form-invariant. The transformations in (A.1) can be formally written in a more symmetical manner as [10]:

$$\beta_A - \beta_B + \beta_C - \beta_A\beta_B\beta_C = 0 \tag{A.4}$$

equivalent to the three transformations:

$$\beta_A = \frac{\beta_B - \beta_C}{1 - \beta_B\beta_C}, \quad \beta_B = \frac{\beta_C + \beta_A}{1 + \beta_C\beta_A}, \quad \beta_C = \frac{\beta_B - \beta_A}{1 - \beta_B\beta_A}. \tag{A.5}$$

Setting $\beta_C = 0$ gives the indentity operator $\beta_A = \beta_B$ while substituting the equation for β_B in that for β_A recovers β_A independently of the value of β_C. The second transformation in (A.5) is therefore the inverse of the first. The transformations in (A.4) or (A.5) are therefore form invariant and contain both identity and inverse operations, so that they constitute a group.

Applying the initial conditions specified above to the first RSTG transformation in (5.10) gives:

$$\tilde{\beta}_u = \frac{\beta_{v'}}{\gamma(v)} + \beta_v, \quad \tilde{\beta}_w = \frac{\beta_{v''}}{\gamma''} + \tilde{\beta}_u. \quad \text{(RSTG)} \tag{A.6}$$

The operational definition of $\tilde{\beta}_u$ is the velocity of the frame S'' observed in S. The quantity $\beta_{v''}/\gamma''$ is the velocity of the frame S''' relative to S'' as observed in the frame S. The velocity, $\beta_{v''}$ of S''' in S'', as fixed by the initial conditions, is scaled by the TD factor γ'' between the frames S'' and S: $\Delta t = \gamma''\Delta t''$[5]. The TD factor γ'' is given, in terms of the fixed velocities v and v' by the transformation equation:

$$V_0'' \equiv \gamma'' = \gamma(v)[\gamma(v') + \beta_v\beta_{v'}\gamma(v')]. \tag{A.7}$$

As can be seen in Fig. 3, this is the Lorentz transformation of the temporal component of the dimensionless 4-vector velocity of the frame S'': $(V_0'; V') \equiv (\gamma(v'); \beta_{v'}\gamma(v'))$, between the frames S' and S. The transformation formula (A.7) follows from that of relativistic energy, (3.10), on dividing through by the factor κm. Combining (A.6) and (A.7) the observed velocity, $\tilde{\beta}_w$ of the frame S''', in the frame S, in terms of the initial velocities v, v' and v'' is:

$$\begin{aligned}
\tilde{\beta}_w &= \frac{\beta_{v''}}{\gamma''} + \frac{\beta_{v'}}{\gamma(v)} + \beta_v \\
&= \frac{\beta_{v''}}{\gamma(v)\gamma(v')[1 + \beta_v\beta_{v'}]} + \frac{\beta_{v'}}{\gamma(v)} + \beta_v.
\end{aligned} \tag{A.8}$$

Comparison of this formula with that for $\tilde{\beta}_u$ in (A.6) shows that RSTG velocity transformations are not form-invariant. Furthermore, applying first the transformation giving $\tilde{\beta}_u$ in (A.6), and then the same transformation with the replacement: $v \to -v$, gives

$$\frac{\beta_{v'}}{\gamma(v)^2} + \beta_v\left(\frac{1}{\gamma(v)} - 1\right) \neq \beta_{v'} \tag{A.9}$$

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[5]Note the different meaning of the symbol γ'' here as compared to that in Eq. (5.2)

which, unlike the same sequence of operations for the RKIN transformation giving $\hat{\beta}_u$ in (A.1), does not yield the inverse transformation. Since the left side of (A.9) does reduce to $\beta_{v'}$ in the Galilean limit $\gamma(v) \to 1$, the breakdown of the group property evident in (A.8) and (A.9) is a purely relativistic, $O(\beta^2)$, effect.

Defining:

$$\beta'' \equiv \frac{\sqrt{(\gamma'')^2 - 1}}{\gamma''} \tag{A.10}$$

and using (A.7) to eliminate γ'' on the right side of this equation it is found that

$$\beta'' = \frac{\beta_v + \beta_{v'}}{1 + \beta_v \beta_{v'}} = \hat{\beta}_u \tag{A.11}$$

which enables (A.8) to be written as:

$$\tilde{\beta}_w = \frac{\beta_{v''}}{\gamma(\hat{u})} + \frac{\beta_{v'}}{\gamma(v)} + \beta_v. \tag{A.12}$$

It is then seen that the RKIN velocity transformation formula in (A.1) enables calculation of the TD factor γ'' that appears in the RSTG transformation which correctly describes velocities observed in different frames in the same space-time experiment. This is a correct, and physically relevant, application of the RKIN velocity transformation.

Dividing the relativistic momentum transformation formula through by the factor $m\sqrt{\kappa}$ yields the Lorentz transformation of the spatial component of the 4-vector velocity $(V_0'; V')$:

$$V'' \equiv \gamma'' \beta'' = \gamma(v)[\gamma(v')\beta_{v'} + \beta_v \gamma(v')]. \tag{A.13}$$

The ratio of Eq. (A.13) to (A.7) recovers Eq. (A.11) above. In fact Eqs. (A.7), (A.11) and (A13) are algebraically equivalent: if any one of the equations is postulated then the remaining two may be derived by purely algebraical manipulation.

It is instructive to compare the Lorentz transformation equations for the the 4-vector velocity $(V_0'; V')$, (A.7) and (A.13):

$$V_0'' = \gamma(v)[V_0' + \beta_v V'], \tag{A.14}$$
$$V'' = \gamma(v)[V' + \beta_v V_0'] \tag{A.15}$$

with the inverse of the conventional space-time LT in (4.23) and (4.24):

$$x_0 = \gamma(v)[x_0' + \beta_v x'], \tag{A.16}$$
$$x = \gamma(v)[x' + \beta_v x_0'] \tag{A.17}$$

where $x_0 \equiv \sqrt{\kappa} t$ and $x_0' \equiv \sqrt{\kappa} t'$. Although the right sides of (A.14) and (A.16) or (A.15) and (A.17) are formally identical, the physical meanings of the equations are completely different. Quantities in three inertial frames, S, S' and S'' are connected by Eqs. (A.14) and (A.15), quantities in only two, S and S' by (A.16) and (A.17). Since x' is, by definition, time-independent, then, as explained in Section 4 above, it necessarily vanishes. In contrast in (A.14) and (A.15) the vanishing of V' requires $v' = 0$ so that

the frames S and S'' are identical. The space-time transformation equations (A.16) and (A.17) are equivalent to the equations:

$$
\begin{aligned}
x_0 &= \gamma(v)x_0', \quad \text{(TD)} & \text{(A.18)} \\
x' &= 0, \quad \text{(Worldline in S')} & \text{(A.19)} \\
x &= \gamma(v)\beta_v x_0' = \beta_v x_0. \quad \text{(Worldline in S)} & \text{(A.20)}
\end{aligned}
$$

No such simplification or physical interpretation exists for the 4-vector velocity transformation equations (A.14) and (A.15).

References

[1] W.Voigt, Goett. Nachr. 1887 p.41.

[2] H.A.Lorentz, Proc. K. Ak. Amsterdam **6** 809 (1904).

[3] J.Larmor, Philos. Trans. R. Soc. London, **190 A** 205 (1897).

[4] H.Poincaré, Rend. del Circ. Mat. di Palermo **21** 129-146, 166-175 (1906).

[5] A. Einstein, Annalen der Physik **17**, 891 (1905). English translation by W.Perrett and G.B.Jeffery in 'The Principle of Relativity' (Dover, New York, 1952) p.37, or in 'Einstein's Miraculous Year' (Princeton University Press, Princeton, New Jersey, 1998) p.123.

[6] W.Ignatowsky, Phys. Zeitschr. **11** 972 (1910).

[7] Frank and H. Rothe, Ann. Phys. **34** 825 (1911).

[8] L.A. Pars, Phil. Mag. **42** 249 (1921).

[9] V.Berzi and V.Gorini, Journ. Math. Phys. **10** 1518 (1969).

[10] J.H.Field, Helv. Phys. Acta **70** 542 (1997), arXiv pre-print: http://xxx.lanl.gov/abs/physics/0410262v1. Cited 27 Oct 2004.

[11] A.Pais:' Subtle is the Lord', Oxford University Press (1982), Ch 7, p.138.

[12] W.Pauli, 'Relativitätstheorie' (Springer, Berlin 2000). English translation,'Theory of Relativity' (Pergamon Press, Oxford, 1958) Section 4, p.9.

[13] J.H. Field, 'The physics of space and time III: Classification of space-time experiments and the twin paradox'. arXiv pre-print: http://xxx.lanl.gov/abs/0806.3671v2. Cited 7 Nov 2011.

[14] J.H. Field, 'Primary and reciprocal space-time experiments, relativistic reciprocity relations and Einstein's train-embankment thought experiment. arXiv pre-print: http://xxx.lanl.gov/abs/0807.0158v2. Cited 11 Nov 2011.

[15] J.H. Field, 'Space-time attributes of physical objects and the laws of space-time physics' arXiv pre-print http://xxx.lanl.gov/abs/0809.4121v1 Cited 24 Sep 2008.

[16] J.H. Field, Fundamental Journal of Modern Physics, Vol **2** Issue 2 139 (2011); arXiv pre-print http://xxx.lanl.gov/abs/1210.2270v1. Cited 8 Oct 2012.

[17] J.H. Field, Fundamental Journal of Modern Physics, Vol **4** Issue 1-2 1 (2012); arXiv pre-print: http://xxx.lanl.gov/abs/1307.7962. Cited 30 Jul 2013.

[18] R.P.Feynman, R.B.Leighton and M.Sands, 'Lectures on Physics, Vol I (Addison-Wesley, Reading, Massachusetts, 1964) Section 15-9.

[19] J.H. Field, Phys. Scripta **73** 639 (2006).

[20] E.J. Post, Rev. Mod. Phys. **39** No 2 475 (1967).

[21] R.D. Klauber, Found. Phys. Lett. **16** (5) 447 (2003).

[22] G. Sagnac, Compt. Rend. Acad. Sci. **157** 708, 1410 (1913).

[23] A.Dufour and F.Prunier, Compt. Rendu. Acad. Sci. **204** 1925 (1937)

[24] F. Selleri, Foundations of Physics Letters, **10** (1) 73 (1997).

[25] J.H. Field, 'The Sagnac effect and transformations of relative velocities between inertial frames'. Paper available at: http://www.relativity-myths.org.uk/jhfield/index.php.

[26] N. Ashby, 'Relativity and the global positioning system', Physics Today, May 2002, pp.41-47.

[27] Y. Saburi, M.Yamamoto and K. Harada, IEEE Trans. Instr. Meas. **25** 473 (1976).

[28] C.-C. Su, Europhys. Lett. **56** (2) 170 (2001).

[29] C.-C. Su, Eur. Phys. J. Su **C 21** 701 (2001).

[30] K. Schwartzschild, Sitzungberichte Prüssiche Akademie der Wissenschaften p.198 (1916).

[31] S. Weinberg, 'Gravitation and Cosmology, Principles and Applications of the General Theory of Relativity', (John Wiley, New York, 1972) Ch 8 Section 2.

[32] J.H. Field, Eur. Phys. J. **C 72** 2191 (2012).