Relativistic relative velocity transformations demonstrate the invariance of length intervals

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Abstract

A thought experiment with two small objects sliding, in opposite directions, along a rotating ring is used to derive, in both Galilean and Special relativity, the transformation formulas for relative angular and spatial velocities. The latter formula requires, in conjunction with time dilation, the invariance of length intervals in Special relativity and resolves the Ehrenfest rotating disc paradox. The inapplicablity, to the problem considered, of the conventional relativistic parallel velocity addition relation is demonstrated, and some correct physical interpretations of the relation are discussed. How spurious and correlated 'length contraction' and 'relativity of simultaneity' effects arise from misuse of the Lorentz transformations is explained. Simple formulas, equivalent to the Lorentz transformations, describing the space-time geometry of a ponderable object in different inertial frames, are presented.

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1 Introduction

The present paper describes a simple thought experiment, where two objects slide, in opposite directions, around a narrow rotating ring, that enables the derivation of transformation formulas for *relative* angular and spatial velocities. The analysis is performed in both Galilean and Special relativity theories. To avoid possible problems related to definitions of coordinate systems, only angular, spatial and temporal intervals, independent of the choice of coordinate system are considered. The transformation formula for angular velocities obtained, previously derived by Post [1], is markedly different from the one given by the conventional parallel velocity addition formula of Special relativity. Application of the corresponding transformation formula for parallel relative spatial velocities shows that length intervals are invariant and that the Reciprocity Principle [2] of Galilean relativity no longer holds in Special relativity. How the spurious 'length contraction' and 'relativity of simultaneity' effects of conventional Special relativity theory arise from misuse of spatial and temporal coordinate systems in the Lorentz transformation equations is also explained.

The structure of the paper is as follows: The following two sections describe analyses of the thought experiment in Galilean and Special relativity, respectively. In Section 4 the invariance of length intervals is derived and the Ehrenfest paradox concerning the ratio of the circumference of a rotating disc to its radius, is resolved. In Section 5 it is shown that application of the conventional parallel velocity addition formula of Special relativity to the thought experiment considered leads to violation of the principle enunciated by Langevin [3] of the frame invariance of a triple world line coincidence. In the concluding section, the previous derivation of the transformation formula, due to Post, for relative angular velocities, is recalled and the correct physical meaning of the conventional parallel velocity addition formula is discussed. Finally, the unphysical origin of the correlated 'length contraction' and 'relativity of simultaneity' effects is discussed and simple space-time equations for the world lines of ponderable objects in different frames, in both Galilean and Special relativity, are presented. The latter, though simpler, are physically equivalent to the (correctly interpreted) space-time Lorentz transformation equations.

2 Galilean-relativistic analysis

The thought experiment considered is shown in Fig. 1. Two identical small circular objects of radius much less than that, ρ', of a narrow ring, slide, in a frictionless manner, around, the interior of ring, in opposite directions, with constant and equal angular speeds relative to the ring. The clockwise and anticlockwise rotating objects are denoted as $(+)$ and $(-)$ respectively, At time $t' = 0$, as registered by a clock stationary in the rest frame, S', of a fixed point on the ring, the objects are aligned with the point O' on the ring. The top left hand configuration in Fig. 1 shows the configuration in S' at time $t' = \phi'_-/\omega' = \phi'_+/\omega'$, where ϕ'_-, ϕ'_+ are angular displacements of the objects from O' and

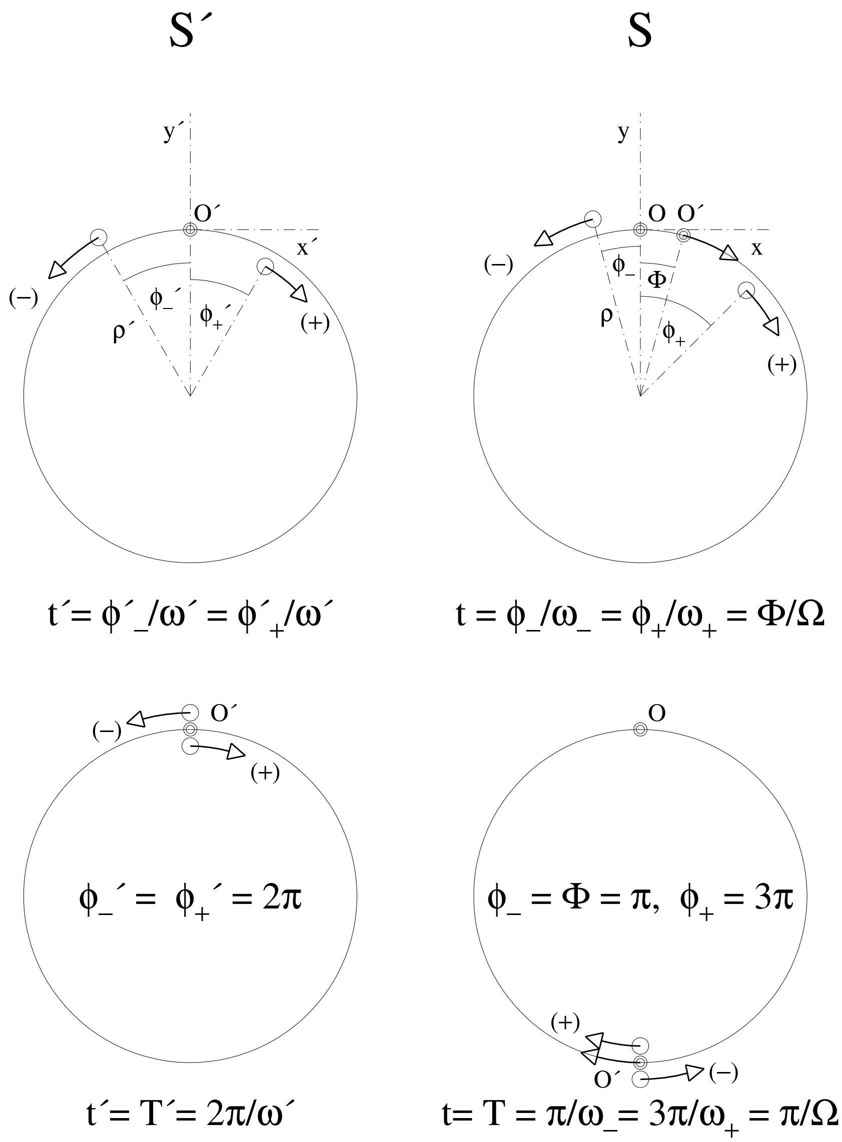

Figure 1: *Two small objects* $(+)$, $(-)$ *slide, in opposite directions, with constant angular velocity,* ω', *relative to a narrow ring of radius* ρ'. *The ring rotates about its centre in a clockwise direction with angular velocity* $\Omega = \omega_-$ *in the inertial frame S. The left-hand figures show configurations in the comoving inertial frame S' of the fixed point O' on the ring, at different times. In the lower-left figure, the objects have completed one circuit of the ring. The right hand figures show the corresponding configurations in the frame S. At* $t = 0$ *the fixed point O in S is aligned with O'. For clarity the sliding objects are slightly displaced radially relative to the ring*

$\omega' = d\phi'_-/dt' = d\phi'_+/dt'$.[1] The ring rotates with constant angular velocity, $\Omega = d\Phi/dt$, in the clockwise direction, in the inertial frame, S, in which the center of the ring is at rest. The time t is recorded by a clock at rest in the frame S. The frame S' is now identified with the comoving inertial frame of O'. In this frame O' is the origin of coordinates, the x' axis is tangent to the ring and the y' axis in the outward radial direction. Similar coordinate axes with origin at O are defined in the frame S (see Fig. 1). At time $t = t' = 0$, the point O' on the ring is coincident with the point O. The top right hand configuration in Fig 1. is in S at time $t = \phi_-/\omega_- = \phi_+/\omega_+$ where $\omega_- = d\phi_-/dt$ and $\omega_+ = d\phi_+/dt$. The bottom left hand configuration is in S' at time $t' = T' = 2\pi/\omega'$ when both objects have completed one circuit of the ring, in opposite directions, and are again aligned with O'. The lower right hand configuration in Fig. 1 is in S at time $t = T = \pi/\Omega = \pi/\omega_- = 3\pi/\omega_+$ for the case where $\Omega = \omega_- = \omega_+/3$. At this time, in the frame S, both O' and the object $(-)$ have rotated through an angle of π rad while the object $(+)$ has rotated through an angle of 3π rad. As is the case in the frame S' at time $t' = T'$ the two objects and O' are aligned in S at time $t = T$.

Introducing the angles ϕ^r_-,ϕ^r_+ which give the angular separation of the objects from O' in the frame S:

$$\phi^r_- \equiv \phi_- + \Phi, \qquad \phi^r_+ \equiv \phi_- - \Phi \tag{2.1}$$

then

$$\frac{d\phi^r_-}{dt} \equiv \omega^r_- = \frac{d\phi_-}{dt} + \frac{d\Phi}{dt} = \omega_- + \Omega \tag{2.2}$$

and

$$\frac{d\phi^r_+}{dt} \equiv \omega^r_+ = \frac{d\phi_+}{dt} - \frac{d\Phi}{dt} = \omega_+ - \Omega. \tag{2.3}$$

Since the lower right hand configuration in Fig. 1 corresponds to $\phi^r_- = \phi^r_+ = 2\pi$ it follows that

$$\omega^r_- = \omega^r_+ = \frac{2\pi}{T} \tag{2.4}$$

In Galilean relativity, where time is the same in all frames of reference, $T = T'$ so that (2.4) gives:

$$\omega^r_- = \omega^r_+ = \frac{2\pi}{T'} = \omega' \tag{2.5}$$

(2.3), (2.4) and (2.5) then give the transformation formula for angular velocities between the frames S and S' as:

$$\omega'_\pm = \omega_\pm \mp \Omega \qquad (\underline{\text{Galilean relativity}}) \tag{2.6}$$

Note that $\omega'_\pm = \omega'$ is the relative angular velocity of the objects and O' in the frame S' while the right side of (2.6) gives the angular velocities of the objects relative to O' in the frame S. Eq. (2.6) is therefore a *relative angular velocity transformation formula*. In Galilean relativity the relative angular velocities of O' and the objects $(+)$ or $(-)$ are equal in the frames S and S'.

3 Special-relativistic analysis

The angles ϕ'_-, ϕ'_+ specifiy the positions of the objects in S', ϕ_-, ϕ_+ and Φ positions in

[1]Note that, with these definitions, all angular velocities and velocities are positive quantities.

S. This is also the case in Special relativity so Fig. 1 requires no modification in this case. However, since $\omega' = d\phi'_-/dt' = d\phi'_+/dt'$ while $\omega_- = d\phi_-/dt$, $\omega_+ = d\phi_+/dt$, $\Omega = d\phi/dt$, the non-universality of time $t \neq t'$ in Special relativity implies that angular velocities in different frames are modified as compared to the Galilean case. Since t' is measured by a clock at rest in S' and t by a clock at rest in S, it is necessary to consider points on the world line of a clock at rest in S' (say at the position O') in comparison with points on the world line of the clock a rest in S. This is conveniently done, without the necessity to introduce specific coordinate systems in the frames, by considering the invariant interval relation for two points on the world line of the point O'. Using polar coordinates appropriate to the geometry of the problem under consideration the relation is:

$$(ds)^2 = c^2(dt')^2 - (dr')^2 - (r'd\phi')^2 = c^2(dt)^2 - (dr)^2 - (rd\phi)^2 \tag{3.1}$$

where primed intervals are defined in the frame S' and unprimed ones in the frame S. Since O' is a rest in S' $dr' = d\phi' = 0$. Also, since for O' in S, $r = \rho =$ constant so that $dr = 0$, the world line of O' in S is (See Fig. 1) the helix: $r = \rho$, $\phi = \Phi = \Omega t$. Inserting these conditions in (3.1) gives:

$$(ds)^2 = c^2(dt')^2 = c^2(dt)^2 - \rho^2\Omega^2(dt)^2 = c^2\left[1 - \frac{\rho^2\Omega^2}{c^2}\right](dt)^2 \tag{3.2}$$

or

$$dt' = dt\sqrt{1 - \frac{\rho^2\Omega^2}{c^2}} \equiv \frac{dt}{\gamma(\rho,\Omega)} \equiv \frac{dt}{\gamma} \tag{3.3}$$

The time dilation effect for points on the rotating ring is then the same as for an inertial frame moving with speed $v = \rho\Omega$ relative to the frame S. The derivation of (3.3) is in agreement with that of Møller [4] for the time dilation effect for a clock on a rotating disc. The transverse acceleration experienced by points on the ring, in the frame S, therefore has no effect on the rate of clocks comoving with them. Using (3.3) the relation $T' = T$ of the Galilean analysis is replaced by:

$$T = \gamma T' \tag{3.4}$$

(3.4) and (2.4) then give:

$$\omega^r_- = \omega^r_+ = \frac{2\pi}{T} = \frac{2\pi}{\gamma T'} = \frac{\omega'}{\gamma} \tag{3.5}$$

so that the relative angular velocity transformation relation (2.6) of Galilean relativity is changed to

$$\omega'_\pm = \gamma(\omega_\pm \mp \Omega) \qquad (\underline{\text{Special relativity}}) \tag{3.6}$$

By considering a configuration with $\Phi = 0$ in Fig. 1 so that The O'x' and Ox axes shown are parallel, the relativistic transformation formula for parallel relative velocities may be derived from (3.6). Since the radius vectors joining the center of the ring to O' in the frame S', and to the corresponding point in the frame S, are perpendicular to the boost direction between the S and S', the corresponding spatial intervals are equal: $\rho' = \rho$. Introducing the space velocities:

$$u'_\pm \equiv \rho'\omega'_\pm = \rho\omega', \qquad u_\pm \equiv \rho\omega_\pm, \qquad v \equiv \rho\Omega \tag{3.7}$$

into (3.6) gives

$$u'_\pm = \gamma_v(u_\pm \mp v) \qquad (\underline{\text{Special relativity}}) \tag{3.8}$$

where $\gamma_v \equiv 1/\sqrt{1-(v/c)^2}$. The formula (3.8) is given below the acronym RRVTR for Relativistic Relative Velocity Transformation Relation.

An interesting special case of (3.8) is $u_- = 0$, in which case the anticlockwise rotating object is a rest relative to O; i.e. u'_- is the speed of O relative to O' in the frame S':

$$u'(\mathrm{O}) = \gamma_v v \tag{3.9}$$

Thus the speed of O relative to O' in S' is $\gamma_v v$, not v, in the case in which the speed of O' relative to O in S is specified to be v. The 'Reciprocity Principle' [2] which holds in Galilean relativity: 'If the velocity of B relative to A is $\vec{v}$ in the rest frame of A then the velocity of A relative to B is -$\vec{v}$ in the rest frame of B', is then invalid in Special relativity.

A example of the application of (3.9) is to high energy muons produced near the top of the Earth's atmosphere that arrive at the surface of the Earth due to the time dilation effect. This is possible not, as in the conventional text book explanation, because the thickness of the atmosphere is reduced by the factor $1/\gamma_v$ by 'length contraction' in the muon rest frame but because the speed of the atmosphere relative to the muon is γ_v times greater than the speed of the muon through the atmosphere.

4 Invariance of spatial intervals in Galilean and Special relativity and resolution of the Ehrenfest paradox

Consider the motion of the clockwise-turning object during small time intervals $\Delta t'$, Δt in S, S' after $t = t' = 0$. Suppose that, as shown in Fig. 2, a small interval of length $\Delta x'$ is marked out on the ring —say there are two shallow grooves prependicular to the plane of the ring separated by this distance. If $\Delta x' \ll \rho$ the motion of the object is essentially parallel to the x, x' axes shown in the upper configurations in Fig. 1. The top left (right) figures in Fig. 2 show configurations in S' (S) at time $t = t' = 0$. The lower left (right) configurations are in S'(S) at $t' = \Delta t'$ ($t = \Delta t$) when the object has just arrived at the end of the marked interval. The space-time geometry of the lower configurations in Fig. 2 gives:

$$\Delta x' = u'_+ \Delta t' \tag{4.1}$$

$$u_+ \Delta t = v\Delta t + \Delta x \tag{4.2}$$

Transposing (4.2) and using (3.8):

$$\Delta x = \Delta t(u_+ - v) = \frac{u'_+ \Delta t}{\gamma_v} = u'_+ \Delta t' \tag{4.3}$$

where in the last member the time dilation relation (3.3) has been used. Combining (4.1) and (4.3) gives

$$\Delta x' = \Delta x \tag{4.4}$$

which demonstrates the invariance of length intervals in both Galilean relativity where $\gamma(v) = 1$, $\Delta t = \Delta t'$ and in Special relativity where $\gamma_v > 1$ and $\Delta t = \gamma_v \Delta t'$.

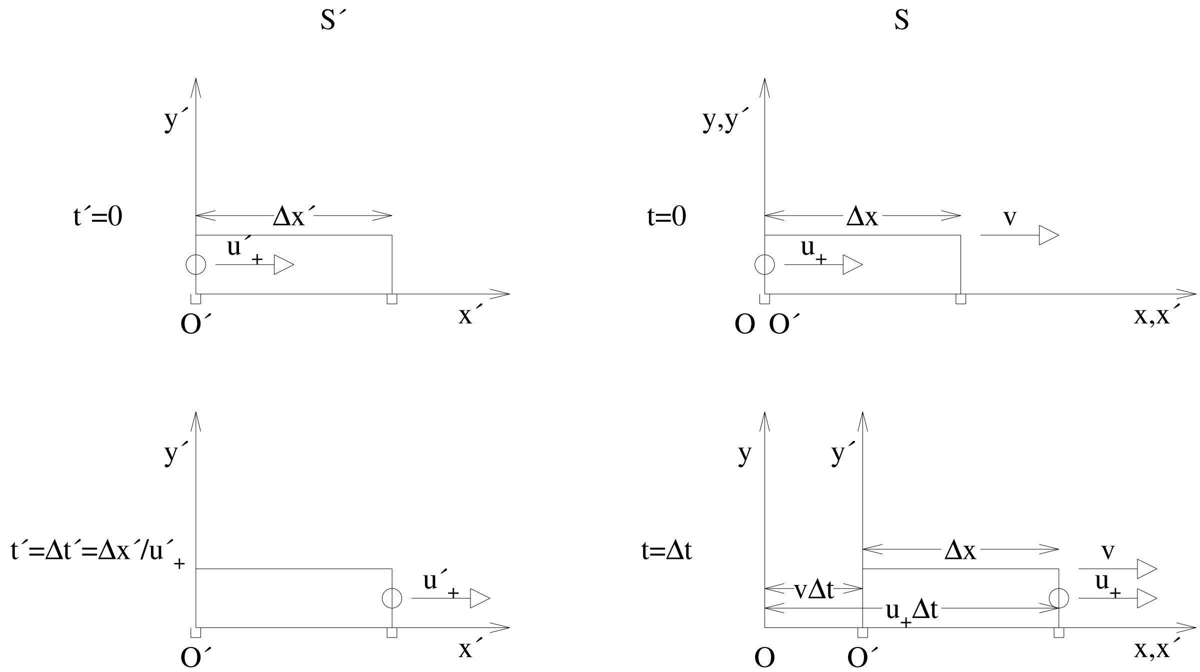

Figure 2: *Configurations of* (+)*, O' and O for small time intervals* $\Delta t'$*,* Δt *such that the O'*x'*, Ox axes shown in Fig. 1 are essentially parallel. The object* (+) *crosses the fixed interval, marked by the small square grooves in the ring,* $\Delta x'$*, in S', in the time interval* $\Delta t'$*. During this interval O' moves the distance* $v\Delta t$ *away from O in the frame S. Configurations at* $t' = 0$*,* $t' = \Delta t'$ *in S' are shown on the left side, and configurations at* $t = 0$*,* $t = \Delta t$ *in S on the right side. The geometry of the lower right figure, the RRVTR (3.8) and the time dilation relation:* $\Delta t = \gamma_v \Delta t'$ *require that* $\Delta x = \Delta x'$*. For clarity the object* (+) *is slightly displaced in the outward radial direction. See text for further discussion.*

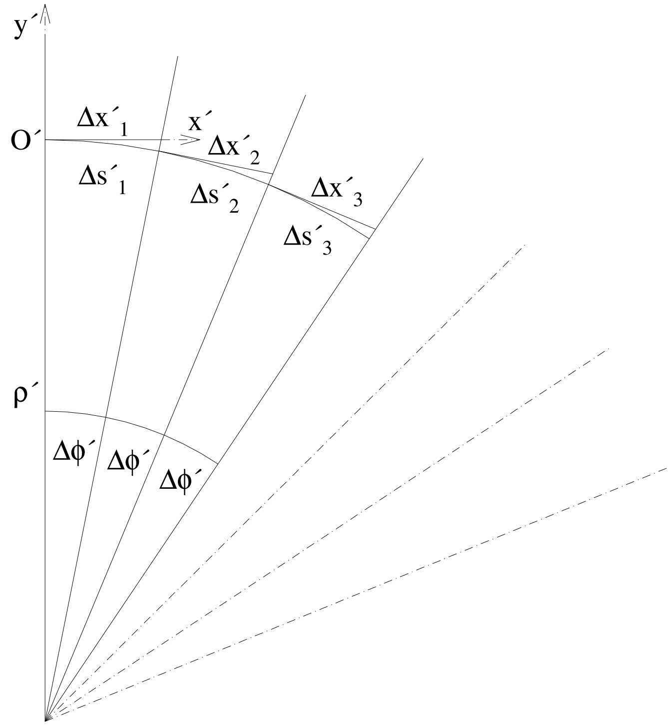

Figure 3: *Division of the ring in the frame S' into n arc segments of length* $\Delta s'_1$, $\Delta s'_2$, $\Delta s'_3$,... *each subtending an angle* $\Delta\phi'$ *at the ring center. The corresponding tangential line segments subtending the same angle are of length:* $\Delta x'_1$, $\Delta x'_2$, $\Delta x'_3$,... .

Suppose that the ring is divided, in the frame S', into n equal arc segments, $\Delta s'_i$, of length $\rho'\Delta\phi'$, as shown in Fig. 3. The tangential line segments of length $\Delta x'_i$ corresponding to each of the arc segments are of length $\rho'\tan\Delta\phi'$. For small values of $\Delta\phi'$ ($\Delta\phi' \ll 2\pi$)

$$\Delta x'_i = \rho'\tan\Delta\phi' = \rho'\frac{\sin\Delta\phi'}{\cos\Delta\phi'} = \rho'\Delta\phi'\frac{\left(1-\frac{(\Delta\phi')^2}{6}+...\right)}{\left(1-\frac{(\Delta\phi')^2}{2}+...\right)} = \Delta s'_i\left(1+\frac{(\Delta\phi')^2}{3}\right) + \mathrm{O}((\Delta\phi')^5) \tag{4.5}$$

Summing over all sectors into which the ring is divided:

$$\sum_{i=1}^{n}\Delta x'_i = \sum_{i=1}^{n}\Delta s'_i\left(1+\frac{(\Delta\phi')^2}{3}\right) \;+\; \mathrm{O}((\Delta\phi')^5) = C'\left(1+\frac{(\Delta\phi')^2}{3}\right) \;+\; \mathrm{O}((\Delta\phi')^5) \tag{4.6}$$

where $C' \equiv \sum_{i=1}^{n}\Delta s'_i$ is the circumference of the ring. It follows from (4.6) that:

$$(Lim\ \Delta\phi' \to 0,\ n \to \infty)\sum_{i=1}^{n}\Delta x'_i = C' \tag{4.7}$$

Performing a similar subdivision of the ring into sectors at a given instant in the frame S, it is similarly found that:

$$(Lim\ \Delta\phi \to 0,\ n \to \infty)\sum_{i=1}^{n}\Delta x_i = C \tag{4.8}$$

where C is the circumference of the ring in the frame S. Since, from (4.4), $\Delta x_i = \Delta x'_i$ then (4.7) and (4.8) give $C = C'$ i.e. the circumference of the ring is the same in the frames S and S'.

This resolves the 'Ehrenfest paradox' [5] concerning the ratio of the circumference of a rotating disc to its radius. Ehrenfest remarked that the circumference of the disc would be subject to length contraction, unlike the radius of the disc which is a length interval perpendicular to the boost direction. In the notation of the present paper then $2\pi\rho < 2\pi\rho'$, by consideration of the circumference of the disc, while $\rho = \rho'$ by consideration of its radius. These are incompatible conditions, hence the paradox.

Einstein's analysis of the problem [6, 7] considered length contraction of measuring rods which are used to determine the circumference of the disc. By comparing measurements of the circumference of the disc with such rods, with the disc either at rest or rotating, and assuming length contraction of the rods (but not of the circumference of the disc) in the rotating case, it was concluded that the ratio of the measured circumference to the measured radius is greater than 2π. Since the length contraction effect does not exist (c.f. (4.4), (4.7) and (4.8) above) the ratio is 2π both in the rest frame of the disc and in a frame in which it is in uniform rotation, so that $2\pi\rho = 2\pi\rho'$, removing the paradox.

In contrast to some text-book presentations of the problem [8, 9] no invocation of non-Euclidian geometry is necessary for the relativistic analysis of the rotating disc problem. In the General-relativistic analyses of [8, 9] time dilation is neglected when transforming to the rotating system of coordinates. The special relativistic discussion of [8] repeats Einstein's argument recalled above.

5 Non-applicabilty to the present problem of the conventional parallel velocity addition relation of Special relativity

Making use of Eqs. (3.7), the conventional Relativistic Parallel Velocity Addition Relation (RPVAR) [10]:

$$u'_\pm = \frac{u_\pm \mp v}{1 \mp \frac{u_\pm v}{c^2}} \tag{5.1}$$

may be used to derive the corresponding transformation relation for angular velocities:

$$\omega'_\pm = \frac{\omega_\pm \mp \Omega}{1 \mp \frac{\rho^2 \omega_\pm \Omega}{c^2}}. \tag{5.2}$$

Algebraic transposition of (5.2) gives[2]:

$$\tilde{\omega}_\pm = \frac{\omega'_\pm \pm \Omega}{1 \pm \frac{\rho^2 \omega'_\pm \Omega}{c^2}} \tag{5.3}$$

to be contrasted with the corresponding transposed version of Eq. (3.6):

$$\omega_\pm = \frac{\omega'_\pm}{\gamma_v} \pm \Omega. \tag{5.4}$$

[2]For clarity the value of $\omega_\pm$ given by fixed values of $\omega'_\pm$ and Ω using (5.3) has a tilde accent to distingish it from the similar quantity given by Eq. (5.4).

The angular velocity of the objects relative to that of O' in the frame S are given, according to (5.3), by relations similar to (2.2) and (2.3):

$$\tilde{\omega}_+^r = \tilde{\omega}_+ - \Omega = \frac{\omega'_+ + \Omega}{1 + \frac{\rho^2 \omega'_+ \Omega}{c^2}} - \Omega = \frac{\omega' \left[1 - \frac{\rho^2 \Omega^2}{c^2}\right]}{1 + \frac{\rho^2 \omega' \Omega}{c^2}}, \tag{5.5}$$

$$\tilde{\omega}_-^r = \tilde{\omega}_- + \Omega = \frac{\omega'_- - \Omega}{1 - \frac{\rho^2 \omega'_- \Omega}{c^2}} + \Omega = \frac{\omega' \left[1 - \frac{\rho^2 \Omega^2}{c^2}\right]}{1 - \frac{\rho^2 \omega' \Omega}{c^2}}. \tag{5.6}$$

Choosing $\omega'_+ = \omega'_- = \omega'$ so that the lower left configuration in Fig. (1) occurs at time $t' = T' = 2\pi/\omega'$, it is found from (5.5) and (5.6) that the objects are predicted to be aligned with O' at different times: T_+ and T_- in the frame S:

$$T_+ = \frac{2\pi}{\tilde{\omega}_+^r} = \frac{2\pi \left[1 + \frac{\rho^2 \omega' \Omega}{c^2}\right]}{\omega' \left[1 - \frac{\rho^2 \Omega^2}{c^2}\right]} = \frac{T'(1 + \beta_{u'} \beta_v)}{1 - \beta_v^2} = T\gamma_v(1 + \beta_{u'} \beta_v), \tag{5.7}$$

$$T_- = \frac{2\pi}{\tilde{\omega}_-^r} = \frac{2\pi \left[1 - \frac{\rho^2 \omega' \Omega}{c^2}\right]}{\omega' \left[1 - \frac{\rho^2 \Omega^2}{c^2}\right]} = \frac{T'(1 - \beta_{u'} \beta_v)}{1 - \beta_v^2} = T\gamma_v(1 - \beta_{u'} \beta_v) \tag{5.8}$$

where $\beta_{u'} \equiv u'/c$, $u' \equiv \rho\omega'$ and $\beta_v \equiv v/c$. so that:

$$\frac{T_+ - T_-}{T} = 2\gamma_v \beta_{u'} \beta_v. \tag{5.9}$$

Choosing parameters in the transformation of Eq. (3.6) such that $\omega_- = \Omega$, as in the configurations shown in Fig. 1, then applying this transformation, gives $\omega' = 2\gamma_v \Omega$. Making the further choice $\beta_v = 1/4$ gives then $\beta_{u'} = 2/\sqrt{15}$. In Fig. 3 the x'- or x-projections of the world lines of the clockwise turning (+) and anticlockwise turning (−) objects, as well as that of the point O', are shown in the frame S' in Fig. 3a and in the frame S as calculated using either Eq. (5.4) in Fig. 3b or Eq. (5.3) in Fig. 3c, with the above choice of β_v and $\beta_{u'}$. In S' the x'-projections of the helical world lines of the (+) and (−) objects are symmetrical sine curves: $x'_\pm = \pm\rho' \sin 2\pi t'/T'$, that of the world line of O' a straight line parallel to the t' axis. In the frame S all the projected world lines are sine curves with different angular frequencies for the (+) and (−) objects, depending on the choice of transformation relations. Setting the radius, ρ', of the ring to unity the following predictions are obtained:

Eq (5.4) (<u>RRVTR</u>)

$$x_- = -\sin \omega_- t = -\sin \frac{\pi t}{T}, \tag{5.10}$$

$$x_+ = \sin \omega_+ t = \sin \frac{3\pi t}{T}, \tag{5.11}$$

$$x_{O'} = \sin \Omega t = \sin \frac{\pi t}{T} \tag{5.12}$$

where, with the chosen parameters (see Fig. 1) $T = \pi/\Omega = \pi/\omega^- = 3\pi/\omega^+ = \gamma_v T' = (4/\sqrt{15})T'$.

Eq (5.3)   (RPVAR)

$$x_- = -\sin\tilde{\omega}_- t = -\sin\frac{\pi\tilde{\omega}_-}{\Omega T}t = -\sin 1.224\frac{\pi t}{T}, \quad (5.13)$$

$$x_+ = \sin\tilde{\omega}_+ t = \sin\frac{\pi\tilde{\omega}_+}{\Omega T}t = \sin 2.715\frac{\pi t}{T}, \quad (5.14)$$

$$x_{\mathrm{O}'} = \sin\Omega t = \sin\frac{\pi t}{T} \quad (5.15)$$

where (5.3) is used to obtain the angular frequency ratios:

$$\frac{\tilde{\omega}_-}{\Omega} = \frac{\beta_{u'}/\beta_v - 1}{1-\beta_{u'}\beta_v} = \frac{8/\sqrt{15}-1}{1-1/(2\sqrt{15})} = 1.224, \quad (5.16)$$

$$\frac{\tilde{\omega}_+}{\Omega} = \frac{\beta_{u'}/\beta_v + 1}{1+\beta_{u'}\beta_v} = \frac{8/\sqrt{15}+1}{1+1/(2\sqrt{15})} = 2.715. \quad (5.17)$$

Notice that the use of the RPVAR instead of the RRVTR increases the rotation frequency in the frame S of the $(-)$ object and decreases that on the $(+)$ object. Using (5.7) and (5.8) with $\beta_{u'} = 2/\sqrt{15}$ and $\beta_v = 1/4$ gives:

$$T_- = 0.8996T, \quad T_+ = 1.166T, \quad (T_+ - T_-)/T = 0.2667.$$

Inspection of Fig. 3 demonstrates the inapplicabilty of the angular velocity transformation formula (5.3), derived from the RPVAR (5.1), to the problem under consideration. If the world lines of O' and the $(+)$ and $(-)$ objects intersect at a common point in the frame S', as is the case at the point P' when $t' = T'$, a similar intersection must occur in all frames of reference. This is indeed the case in Fig. 3b at the point P where $t = T = \gamma_v T'$ (see also Fig. 1) but not in Fig. 3c where the world lines of the $(+)$ and $(-)$ objects intersect with that of O' at the different points P_+, P_- on the x-projection of the world lines. The frame invariance of triple world line intersections may be considered to be the 'zeroth theorem' of space-time physics —a condition that must necessarily be respected by any candidate theory. This is a point that was clearly and graphically stated by Langevin in the 'twin paradox' paper [3]:

> With the new concepts, only one case remains and must remain where a change of coordinate system has no effect. This is the case where two events coincide in both space and time. Such a double coincidence must have, indeed, an absolute meaning because it corresponds to contiguity of the two events and that contiguity may produce a physical phenomenon, a new event, which has necessarily an absolute sense. Consider again the previous example. If the two objects which leave the cart by the same hole do so simultaneously, if their departures coincide both in space and time, there may result a collision which breaks the objects, and that collision has an absolute sense in that, in no possible conception of the world, electromagnetic or mechanical, such a spacetime coincidence, if it exists for one group of observers, could it not exist for another, whatever their motion relative to the first. For those that see the cart pass by as well as for those inside it, the two objects have broken each other because they passed at the same time at the same point.[3]

[3]This English translation of French text of [3] is taken from the present author's paper [11].

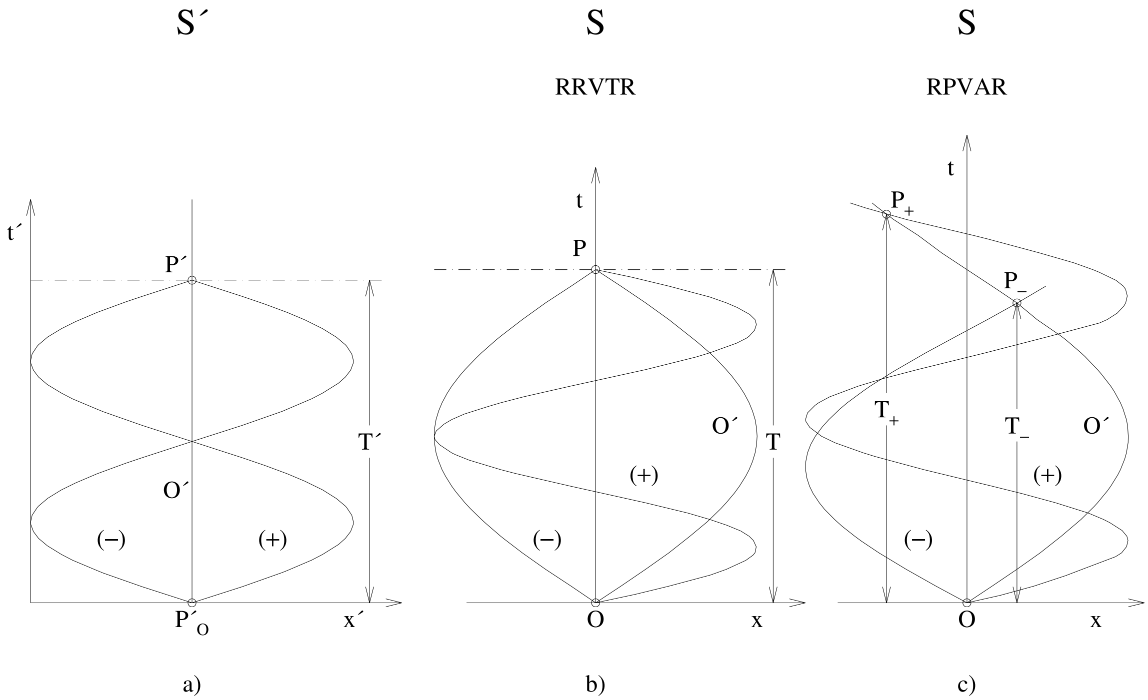

Figure 4: *x'- or x-projections of the world lines of $(-)$, $(+)$ and O' with $\beta_v = 1/4$ and $\beta_{u'} = 2/\sqrt{15}$. a) in the frame S'; b) in the frame S with angular velocities transformed according to Eq. (5.4) (the RRVTR); c) in the frame S with angular velocities transformed according to Eq. (5.3) (the RPVAR). The points P'_0, P', O, P correspond to triple world line coincidences ($(-)$, $(+)$ and O' at the same place at the same time) The points P_+, P_- are intersections of the world lines of $(+)$ and O' only, $(-)$ and O' only, respectively ($(+)$ and O' at the same place at the same time, but not $(-)$, or $(-)$ and O' at the same place at the same time, but not $(+)$.) Because triple world line coincidences are frame-invariant the configuration of c) is physically impossible, given the configuration shown in a). This shows that the transformation formula (5.3) is inapplicable to the problem in hand. See text for further discussion.*

In Fig. 3 the world lines of Langevin's two objects correspond to those of the (+) and (−) objects, that of the hole in the floor of the cart to the world line of O'.

More recently a similar assertion was made by Mermin in a paper [12] in which the RPVAR was derived from the relativity principle and some simple assumptions of symmetry and continuity, without consideration of light signals:

> We consider a race taking place within a long straight train (...). A tortoise and a hare start at the rear of the train toward the front. The hare gets there first, turns immediately around and racing back towards the rear, encounters the tortoise still making its way towards the front. Let u be the speed of the tortoise in the train frame and s, the speed (in either direction) of the hare.
>
> The part of the train where the two meet again is behind the front end by some fraction r of the full length of the train. That fraction is a frame-independent invariant, since there can be no disputing where on the train the meeting occured. (They might for example meet in the 73rd car from the front of a train consisting of 100 identical cars, giving the value $r = 0.73$. Only passengers in car 73 would testify to having observed the encounter, and this testimomy would be acceptable to observers in any frame of reference, even though they might have quite different ideas about the lengths of the cars.)

Here the triple world line intersection is that between those of the tortoise, the hare and the position in the train at which they meet.

It must not be concluded from the non-applicabilty of the RPVAR to the problem considered here that this relation is wrong in all circumstances. As explained in more detail below, it does give correctly the transformation of the *kinematical properties* energy and momentum of a moving object, as well as of the time-dilation factor γ, between different inertial frames. However, it must not be assumed that it gives correctly the transformation of the relative velocity between objects as observed in two inertial frames in the same space-time experiment, which is, instead, given by the RRVTR. The RPVAR is therefore not wrong, but inappropriately applied to the analysis of the the experiment considered here.

6 Discussion and conclusions

The transformation formula (3.6) for relative angular velocities has been previously derived by Post in a review article about the Sagnac effect [1]. Post relativistically generalised an earlier calculation of Langevin [13] to obtain the relation (Eq. (24) of [1]), in the notation of the present paper:

$$d\phi = d\phi' + \gamma\Omega dt'. \tag{6.1}$$

Division throughout by dt' and use of the time dilation relation $dt = \gamma dt'$ gives

$$\frac{d\phi}{dt'} = \gamma \frac{d\phi}{dt} = \gamma\omega_+ = \frac{d\phi'}{dt'} + \gamma\Omega = \omega'_+ + \gamma\Omega \tag{6.2}$$

or, rearranging:

$$\omega'_+ = \gamma(\omega_+ - \Omega) \tag{6.3}$$

which is Eq. (3.6). In the derivation of (6.1) Post considered the simplified Sagnac experiment shown in Fig. 8 of [1]. The geometrical configuration is similar to that shown in Fig. 1 above. However, in the Sagnac experiment, the light signals rotate with angular speed c/ρ in both directions in the fixed inertial frame S. The observed Sagnac effect is a consequence of the different angular velocities, and therefore different path lengths, in the rotating frame, of the counter-rotating light signals, as given by Eq. (3.6) with $\omega_\pm = c/\rho$. Notice that if the angular velocity transformation is performed according to the conventional formula (5.2) with $\omega_\pm = c/\rho$ it is found that $\omega'_+ = \omega'_- = c/\rho$, the path lengths of the signals in the rotating frame are equal and the experimentally-verified Sagnac effect is predicted to vanish!

Clearly the RPVAR (5.1) in inapplicable both to the experiment described in the present paper as well as to the similar Sagnac experiments where the space-time effects observed are governed by the transformation formula (3.6) of relative angular velocities, or the correponding formula (3.8), the RRVTR, for parallel relative spatial velocities. The correct physical interpretation of the RPVAR, also discussed in [14, 15] will now be considered.

Introducing the further notation $\gamma_u \equiv 1/\sqrt{1-\beta_u^2}$, $\beta_u \equiv u/c$ the RPVAR may be written in three mathematically equivalent ways:

$$\begin{aligned} \beta_u &= \frac{\beta_v + \beta_{u'}}{1 + \beta_v \beta_{u'}}, & (6.4) \\ \gamma_u &= \gamma_v \gamma_{u'}(1 + \beta_v \beta_{u'}), & (6.5) \\ \gamma_u \beta_u &= \gamma_v \gamma_{u'}(\beta_v + \beta_{u'}). & (6.6) \end{aligned}$$

These three formulas are algebraically equivalent i.e. if any one of them is postulated, the other two may be obtained by purely algebraic manipulation of the first. It is the versions (6.5) and (6.6) which have a similar structure to that of the space-time Lorentz transformation equations that have the most transparent physical interpretations.

The formula (6.5) gives the transformation of time dilation factor between the frames S' (where the factor is $\gamma_{u'}$) and the frame S (where the factor is γ_u). In this case the moving clock in the time dilation experiment has the velocity u' in the frame S' but is observed in the frame S. The fixed initial parameters of the problem are the velocities v and u'. A direct application of this is to the analysis of special-relativistic effects in the Hafele-Keating experiment [16, 17, 18, 19] where the frame S is the non-rotating Earth-Centered-Inertial (ECI) frame in which 'coordinate time' t is specified, u' is the speed of a clock-carrying aircraft relative to the surface of the Earth and $v = \Omega_{\mathrm{E}} R_\perp$, where $R_\perp$ is the distance from a clock at rest on the Earth's surface to the axis of rotation of the Earth, and Ω_{E} is the angular velocity of rotation of the Earth. Choosing positive or negative values of u' (i.e. considering flights in the West-East or East-West directions respectively) (6.5) predicts the asymmetric time-dilation effect between Earth-bound and airborne clocks that was verified [18, 19] in the Hafele-Keating experiment. The experiment demonstrated conclusively that the time dilation effect between the clocks did not depend only on the relative velocity u' of the two clocks but was given by (6.5) for an airborne clock and by (3.3) for the Earth-bound one.

Introducing the relativistic energy and momentum in the frames S and S' of an object, of Newtonian mass m, according to the relations [20]:

$$\begin{aligned} E & \equiv \gamma_u mc^2, \quad p \equiv \gamma_u \beta_u mc, & (6.7) \\ E' & \equiv \gamma_{u'} mc^2, \quad p' \equiv \gamma_{u'} \beta_{u'} mc & (6.8) \end{aligned}$$

enables (6.5) and (6.6) to be written as:

$$\begin{aligned} E & = \gamma_v (E' + c\beta_v p'), & (6.9) \\ p & = \gamma_v (p' + \frac{\beta_v}{c} E') & (6.10) \end{aligned}$$

which are the well-known Lorentz transformation formulas for relativistic energy and momentum. They follow directly from (6.5) and (6.6) by multiplying throughout by the constant factor mc^2 and using the definitions in (6.7) and (6.8). Note that the velocity parameters v, u, u' which appear in (6.5) and (6.6) are all defined as the velocities of objects in particular inertial frames (v and u in S and u' in S'), and no consideration is given to the observation of *relative velocities* of different objects in the same inertial frame which are described instead by the RRVTR (3.8). Indeed relativistic energy and momentum are attributes of a single physcial object that are always defined in a particular inertial frame. It is clear that the relative velocity, which is an attribute of two distinct physical objects, in a particular frame, cannot enter in any way into the definition of the energy or momentum of the objects in this frame. A 'relative velocity' between two objects is a meaningful physical concept in any reference frame but not 'relative energy' or 'relative momentum' of the objects.

It is interestimg to note that, as previously pointed out in the Appendix of [1], the RRVTR, unlike the RPVAR, does not have mathematical group properties. The inverse of the transformation:

$$\beta_{u'} = \gamma_v(\beta_u - \beta_v) \qquad (6.11)$$

is

$$\beta_u = \frac{\beta_{u'}}{\gamma_v} - \beta_v \qquad (6.12)$$

which has a different algebraic structure. The inverse of the RPVAR

$$\beta_{u'} = \frac{\beta_u - \beta_v}{1 - \beta_u \beta_v} \qquad (6.13)$$

is

$$\beta_u = \frac{\beta_{u'} + \beta_v}{1 + \beta_{u'} \beta_v} \qquad (6.14)$$

that has a similar algebraic structure resulting in a group property and is given by the replacements $v \to -v$, $u' \leftrightarrow u$, i.e. relabelling of variables, in (6.13). Making the same replacements in (6.11) gives not the inverse (6.12) but instead the unphysical equation $\beta_u = \gamma_v(\beta_{u'} + \beta_v)$. Axiomatic derivations such as those used to derive the RPVAR [12] or the Lorentz transformations [21], by postulating a group property, cannot, therefore, be used to derive the RRVTR.

Only coordinate-frame independent angular, length and time *intervals* have been considered in the calculations presented above. In the case that the frame S' is a time-independent rather than a comoving inertial frame, corresponding to the limits $\rho \to \infty$, $\Omega \to 0$, $v = \rho\Omega =$ constant of these calculations, the world line of an object at rest in the frame S' can be written, in S' and S respectively, as

$$x' = L, \qquad x(t) = vt + L. \tag{6.15}$$

Here particular spatial coordinate systems have been introduced in S' and S. From the world line equation in S it can be seen that $L \equiv x(t = 0)$, while the freedom of choice of coordinate origin in S' has been used to set $x' = x(t = 0) \equiv L$. The world line equations in (6.15) are the same in both Galilean and Special relativity. In Galilean relativity time is universal, $t' = t$, whereas in Special relativity, with a particular choice of clock calibration constants the coordinate-independent time dilation relation for time intervals $\Delta t = \gamma_v \Delta t'$ can be written in terms of synchronised clock settings (epochs) in the frame S, (S') t (t') as

$$t = \gamma_v t'. \tag{6.16}$$

Equations (6.15) and (6.16) provide a complete description, within a particular choice of space and time coordinate systems, of the space-time geometry of an arbitary object at rest in the frame S' and observed from the frame S. In Galilean relativity the world lines in (6.15) are unchanged and the time dilation relation (6.16) is replaced by $t' = t$.

The world line equation of the object in S and the time dilation relation (6.16) may be combined in the following manner:

$$t' = \frac{t}{\gamma_v} = \frac{t\gamma_v}{\gamma_v^2} = \gamma_v\left[t - \frac{v^2 t}{c^2}\right] = \gamma_v\left[t - \frac{v(x(t) - L)}{c^2}\right]. \tag{6.17}$$

This is the Lorentz transformation of time between the frames S and S'. Combining the world lines equations in the frames S and S' in (6.15) into a single equation as:

$$x' - L = \gamma_v[x(t) - L - vt] = 0 \tag{6.18}$$

gives the space Lorentz transformation equation corresponding to the time transformation (6.17). Note however that the factor γ_v in (6.18) is discretionary. The formula is equivalent to the two world line equations in (6.15) when γ_v is replaced by an arbitary finite function of v/c. The standard Lorentz transformation, as derived by Einstein [10] corresponds to a particular choice of coordinate system in the frame S such that $L \equiv x(t = 0) = x' = 0$. i.e. the object is placed at the coordinate origins of both S and S' when $t = 0$.

Consider now two objects at different fixed positions in S', $x' = L_1, L_2$ with the same coordinate systems in S and S' as in (6.17) and (6.18), (i.e. $t = t' = 0$ when $x_1 = L_1$, $x_2 = L_2$) to give the transformation equations:

$$t'_1 = \gamma_v\left[t_1 - \frac{v(x_1(t_1) - L_1)}{c^2}\right], \tag{6.19}$$

$$\begin{aligned} t'_2 &= \gamma_v\left[t_2 - \frac{v(x_2(t_2)-L_2)}{c^2}\right], & (6.20)\\ x'_1 - L_1 &= \gamma_v[x_1(t_1) - L_1 - vt_1], & (6.21)\\ x'_2 - L_2 &= \gamma_v[x_2(t_2) - L_2 - vt_2]. & (6.22) \end{aligned}$$

These equations show that the events, on the world lines of the objects, where $x_1 = L_1$ and $x_2 = L_2$, are simultaneous in both S and S': $t_1 = t_2 = t'_1 = t'_2 = 0$. Also since $t_1 = \gamma_v t'_1$ and $t_2 = \gamma_v t'_2$, it follows that when $t'_1 = t'_2 = t'$ then also $t_1 = t_2 = t$, for all values of t'. Simultaneous events on the world lines of the objects in the frame S are also simultaneous in S — there is no 'relativity of simultaneity' (RS) effect. Setting $t_1 = t_2 = t$ in (6.21) and (6.22) i.e. considering the spatial positions of the objects at the same instant in the frame S, it follows from these equations that, for arbitrary and independent values of the parameters of v, γ_v: :

$$\Delta x \equiv x_2(t) - x_1(t) = L_2 - L_1 = x'_2 - x'_1 \equiv \Delta x' \qquad (6.23)$$

consistent with Eq.(4.4) above. Length intervals are the same in the frames S and S' —there is no 'length contraction' (LC) effect.

In the standard Lorentz transformation equations, the object is placed at the origin of coordinates in S', i.e. $L = 0$ in (6.15) and (6.16):

$$\begin{aligned} t'(x'=0,t) &= \gamma_v\left[t - \frac{vx(x'=0,t)}{c^2}\right] & (6.24)\\ x' &= \gamma_v[x(x'=0,t) - vt] = 0 & (6.25) \end{aligned}$$

The spurious RS and LC effects of conventional Special relativity theory occur when (6.24) and (6.25) instead of (6.17) and (6.18) are assumed to be also valid for an object not at the origin of coordinates in S', i.e. when $x' = L' \neq 0$:

$$\begin{aligned} t'(x'=L',t) &= \gamma_v\left[t - \frac{vx(x'=L',t)}{c^2}\right] & (6.26)\\ x' = L' &= \gamma_v[x(x'=L',t) - vt] \neq 0 & (6.27) \end{aligned}$$

Subtracting (6.25) from (6.27) gives

$$L' = \gamma_v[x(x'=L',t) - x(x'=0,t)] \qquad (6.28)$$

or

$$x(x'=L',t) - x(x'=0,t) \equiv \Delta x = \frac{L'}{\gamma_v} = \frac{\Delta x'}{\gamma_v} \qquad (6.29)$$

which is the LC effect. Subtracting (6.24) from (6.26) and using (6.29), gives on rearranging

$$t'(x'=L',t) = t'(x'=0,t) - \frac{\gamma_v v \Delta x}{c^2} \qquad (6.30)$$

which is the RS effect, since the fixed epoch t in the frame S corresponds to different epochs, in the frame S', for objects at $x' = 0$ and $x' = L'$.

The correct transformation equations for an object at $x' = L'$, (6.17) and (6.18) with $L = L'$, differ from the incorrect ones (6.26) and (6.27) by additive constants on the right sides of the equations. That is the former equations may be written:

$$
\begin{aligned}
t' &= \gamma_v \left[t - \frac{vx(t)}{c^2}\right] + T, && (6.31) \\
x' &= \gamma_v [x(t) - vt] + X && (6.32)
\end{aligned}
$$

where

$$
\begin{aligned}
T &= \frac{\gamma_v v L'}{c^2} = \frac{\gamma_v v \Delta x}{c^2}, && (6.33) \\
X &= L'(1 - \gamma_v). && (6.34)
\end{aligned}
$$

It can be seen that the RS term on the right side of (6.30) is exactly cancelled by the constant T in the correct time transformation equation (6.17) or (6.31). The necessity to include the constants T, X to correctly describe the world lines of objects at different fixed positions in the frame S' was clearly stated by Einstein, in the original Special relativity paper [10], immediately after the derivation of the 'standard' Lorentz transformation equations (6.24) and (6.25):

> Macht man über die Anfanslage des bewegten Systems und über den Nullpunkt von τ keinerlei Voraussetzung, so ist auf den rechten Seiten dieser Gleichungen je eine additive Konstante zuzufügen

or, in English:

> If no assumption whatever be made as to the initial position of the moving system and as to the zero point of τ an additive constant is to be placed on the right side of these equations

The quantity τ is t' in the notation of the present paper. This injunction was not however followed, to the writer's best knowledge, by Einstein, or any other author, until the work reported in [22].

The conclusions of this paper for space-time geometry in flat space and the transformation of relative velocities between inertial frames are as follows:

Galilean relativity

$$x'_1 = \chi_1, \quad x_1(t_1) = vt_1 + \chi_1 \ ; \ x'_2 = \chi_2, \quad x_2(t_2) = vt_2 + \chi_2. \qquad (6.35)$$

$$t'_1 = t_1 \quad ; \quad t'_2 = t_2. \qquad (6.36)$$

Special relativity

$$x'_1 = \chi_1, \quad x_1(t_1) = vt_1 + \chi_1 \ ; \ x'_2 = \chi_2, \quad x_2(t_2) = vt_2 + \chi_2. \qquad (6.37)$$

$$t'_1 = \gamma_v t_1 \quad ; \quad t'_2 = \gamma_v t_2. \qquad (6.38)$$

The world line equations in (6.35) and (6.37) are the same in Galilean and Special relativity. As shown above, the world line equations of an object in S and S' are equivalent to the space Lorentz transformation equation and the time transformation equation follows (see Eq. (6.17) above) on combining the world line equation in the frame S with the time dilation relation in (6.38). Considering simultaneous events on the world lines of objects at rest in S': $t'_1 = t'_2$ it can be seen from (6.36) or (6.38) that $t_1 = t_2$ so that the events are simultaneous in S —there is no RS effect. It also follows in both Galilean and Special relativity from (6.35) or (6.37) that

$$x'_2 - x'_1 \equiv L' = \chi_2 - \chi_1 = x_2(t) - x_1(t) \equiv L. \tag{6.39}$$

Spatial separations of objects are therefore invariant as shown in Eq. (4.4) above and there is no LC effect. How the latter, as well as the correlated RS effect, arise from misuse of the Lorentz transformation equations have been explained above.

If two objects move at speeds v and u, parallel to the x-axis in the frame S, the velocity of the object with speed u, in the rest frame of the object with speed v, u', which is the relative velocity of the two objects in the frame S', is given by the relations:

Galilean relativity

$$u' = u - v. \tag{6.40}$$

Special relativity

$$u' = \gamma_v(u - v). \tag{6.41}$$

In Galilean relativity, but not in Special relativity, the magnitude of relative velocity of the two objects is the same in the frames S and S', whereas. in Special relativity, the magnitude of the relative velocity is greater by the factor γ_v in the frame S'. An important special case of (6.41) is $u = 0$, i.e. if the magnitude of the relative velocity between objects at rest, one in the frame S', the other in the frame S, is v in the frame S , then it is also v in the frame S' in Galilean relativity , but instead it is $\gamma_v v$ in Special relativity. The Reciprocity Principle [2] which is tacitly assumed to hold in Einstein's original Special relativity paper [10] and in the subsequent literature and text books on relativity is therefore invalid in Special relativity.

As shown above, the conventional RPVAR is physically equivalent to the transformation laws of relativistic energy and momentum (6.9) and (6.10). These equations transform a kinematical configuration of a *single physical object* in one frame into that of a *single physical object* in another frame. This transformation has no relevance to the transformation of the relative velocity of *two distinct physical objects* from one inertial frame to another, which is correctly given, not by the RPVAR of Eq. (5.1), but by the RRVTR of Eq. (3.8). This assertion is a necessary consequence of the analysis of the thought experiment considered in the present paper and is experimentally confirmed by the existence of the Sagnac effect [1].

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