Ruler measurements give space-time-transformation-independent invariant lengths

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Abstract

Two thought experiments are described in which ruler mesurements of spatial intervals are performed in different reference frames. They demonstrate that such intervals are frame-independent as well as independent of the nature of the space-time transformation equations. As explained in detail elsewhere, the 'length contraction' effect of conventional special relativity theory is therefore spurious and unphysical.

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The concept of a 'ruler measurement' of a spatial interval is a very familiar one in the everyday world. A 'ruler' is a flat object with a rectilinear boundary furnished with equally spaced 'marks', M, specifying positions along the boundary, labelled by ordinal numbers: M(I), $I = 1, 2, 3, ...$. In order to perform a length measurement, 'pointers' on the objects, the spatial separation of which is to be determined, are placed against the ruler and the closest marks to the pointers M(I), M(J) ($J > I$) are noted. The ruler measurement of the spatial separation of the pointers is then $J - I$ in units of the inter-mark separation, with a maximum uncertainty of one unit. A mathematical calculus to rigorously specify such measuremenrs, in terms of pointer-mark coincidences (PMC), was proposed in Ref. [1], but for the purposes of the present paper the simple definition just given is sufficient.

In the case that the to-be-measured objects are at rest relative to the ruler, time plays no role in the measurement. However if they are in uniform or accelerated motion, with the same or different velocities parallel to the edge of the ruler, the times at which the PMCs, constituting the raw measurements, are observed are important. Two distinct situations are possible:

(1) The objects have different velocities.

(2) The objects have the same (possibly time-varying) velocity at all times.

In the first case, the spatial interval between the objects changes with time and a measurement of the spatial separation at any epoch [a] requires the *simultaneous* recording of two PMCs. In case (2), which is that of the examples to be discussed in this paper, simultaneous observation of PMCs is also required to define the spatial separation of the

[a] The word 'epoch' denotes a particular instant of time in some reference frame. It can be operationally defined as a particular PMC corresponding to a spatial coincidence between the moving hand of an analogue clock at rest in the frame (the pointer) and a mark on its dial.

objects, but the latter is a time-independent quantity. Note that the spatial separation in case (2) is a constant —independent of both time and the velocity of the objects— provided that the latter is the same at all epochs. This statement is valid for both uniform and accelerated motion of the two objects.

Using the above definitions it will now be shown, using two simple examples, that ruler measurements of the spatial separation of two objects undergoing similar motion give the same result in both an inertial frame from which their motion is observed, or in the co-moving reference frame of the two objects. Furthermore, this equality is independent of the transformation equations relating space and time coordinates in the two frames. A corollary is that the 'length contraction' and 'relativity of simultaneity' effects of conventional special relativity are illusory [2, 3, 4, 5].

The first example is the thought experiment shown in Fig. 1. Two 'pointer trams', PT1 and PT2, are at rest on straight tracks aligned with two mark poles M3 and M4 respectively, separated by the distance L. The ruler measurement of the initial separation of the trams is thus L. Two further mark poles, M1 and M2, are displaced from M3 and M4 respectively by a distance $3L$ in the direction of motion of the trams. All of the mark poles are equipped with lamps. M3 and M4 flash when the trains start to move, in an identical manner, down the tracks while M1 and M2 flash at the epoch when the front ends of the moving trams are aligned with them. The simultaneity of the signals of the lamps on M1 and M2 or M3 and M4 endorses the validity of the corresponding ruler measurements.

Configurations observed in the rest frame, S, of the mark poles are shown in Figs. 1a and 1b. At $t = 0$, as recorded by a clock at rest in S, the lamps M3 and M4 flash and PT1 and PT2 start to move towards M1 and M2. At time $t = T$, when the instantaneous velocity of PT1 and PT2 is $v(T)$, the front of PT1 is aligned with M1 and the front of PT2 is aligned with M2. At this instant the lamps on M1 and M2 flash. The measured spatial separation of the two trams is then L —the same as when they were at rest— demonstrating the time-independence of their separation for case (2).

The same journey of the trams between the mark poles, as observed in the co-moving frame S' of the trams, is shown in Figs. 1a and 1c. If t' is the epoch recorded by a clock at rest in the frame S', then at $t = t' = 0$, S' is identical to S, so the configuration is again that of Fig. 1a. Observers at rest in the trams will see the mark poles M1 and M2 start to move simultaneously towards them, with identical motion, so that at the time $t' = T'$, when their observed velocity is $v'(T')$, they arrive simultaneously at the front ends of PT1 and PT2 respectively when their lamps flash (Fig 1c). For this ruler measurement, showing that the separation of M1 and M2, as measured in S', is also L, the roles of the trams and mark poles are inverted. The ends of the trams constitute the marks and the moving poles the pointers. It is evident that the flashes of the lamps on M1 and M2 are simultaneous in both S and S' —there is no 'relativity of simultaneity' effect

The conclusions of the thought experiment —the invariance of the results of ruler measurements of the spatial separations of PT1 and PT2 or M1 and M2 performed in different frames in relative motion and the absence of any relativity of simultaneity effect— follow only from the initial postulate of identical motion of the trams in the frame S, and are independent of the form of the space-time transformation equations relating t' to t. Suppose the acceleration of PT1 or PT2 is some arbitary function of t: $a(t)$. The

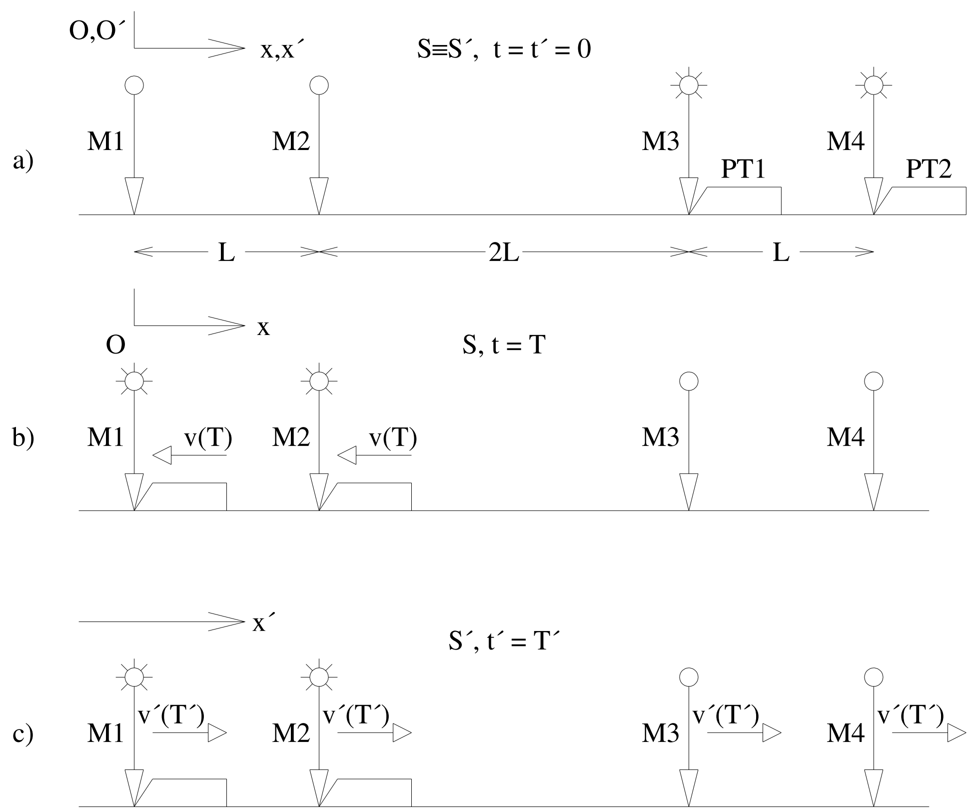

Figure 1: *Spatial configurations in frames S [a),b)] and S'[a),c)] of a thought experiment in which the pointer trams PT1 and PT2 are simultaneously accelerated in a similar manner so as to pass from positions of spatial coincidence with stationary mark poles M3 and M4 at* $t = t' = 0$ *[a)] to simultaneous spatial coincidence with M1 and M2 at time T in S [b)] or time T' in S' [c)]. Time coincidence events are signalled by simultaneous flashes of lamps on M3 and M4 in a) and on M1 and M2 in b) and c). See text for discussion.*

velocity at epoch t of PT1 or PT2 is then

$$v(t) = \int_0^t a(t')dt' \tag{1}$$

and the spatial displacement, $d(t)$, at epoch t is

$$d(t) = \int_0^t v(t'')dt'' = \int_0^t dt'' \int_0^{t''} a(t')dt' \tag{2}$$

The spatial separations $\Delta x_{PT}(t)$ and $\Delta x'_{PT}(t')$ in S and S' respectively are, from the geometry of Fig. 1:

$$\begin{aligned} \Delta x_{PT}(t) &\equiv x(\mathrm{PT2}) - x(\mathrm{PT1}) \\ &= [4L - d(t)] - [3L - d(t)] \\ &= L = \Delta x_{PT}(0) = \Delta x'_{PT}(t') \end{aligned} \tag{3}$$

Space-time transformation equations yield from $a(t)$ the corresponding acceleration $a'(t')$ of M1 and M2 as observed in the frame S'. Similarly to (1) and (2) above it follows that

$$v'(t') = \int_0^{t'} a'(t''')dt''' \tag{4}$$

$$d'(t') = \int_0^{t'} v'(t'')dt'' = \int_0^{t'} dt'' \int_0^{t''} a'(t''')dt''' \tag{5}$$

The spatial separations $\Delta x'_M(t)$ and $\Delta x_M(t)$ of M1 and M2 in S' and S are:

$$\begin{aligned} \Delta x'_M(t') &\equiv x'(\mathrm{M2}) - x'(\mathrm{M1}) \\ &= [L + d'(t')] - d'(t') \\ &= L = \Delta x'_M(0) = \Delta x_M(t) \end{aligned} \tag{6}$$

The world lines of PT1,PT2,M1 and M2 in S and S' for different acceleration programs and space-time transformation equations are shown in Fig. 2 for the following examples:

(i) 'Parabolic motion' in special relativity [6, 7, 8, 9, 10]:

$$x_{PT1}(t) = x_{PT2}(t) - L = 3L - \frac{c^2}{a_0}\left[\sqrt{1 + \left(\frac{a_0 t}{c}\right)^2} - 1\right] \tag{7}$$

$$x'_{M1}(t') = x'_{M2}(t') - L = \frac{c^2}{a_0}\left[\cosh\frac{a_0 t'}{c} - 1\right] \tag{8}$$

(ii) Constant acceleration in Galilean relativity:

$$x_{PT1}(t) = x_{PT2}(t) - L = 3L - \frac{1}{2}a_0 t^2 \tag{9}$$

$$x'_{M1}(t) = x'_{M2}(t) - L = \frac{1}{2}a_0 t'^2 \tag{10}$$

These world lines are the $c \to \infty$ limits of (7) and (8)

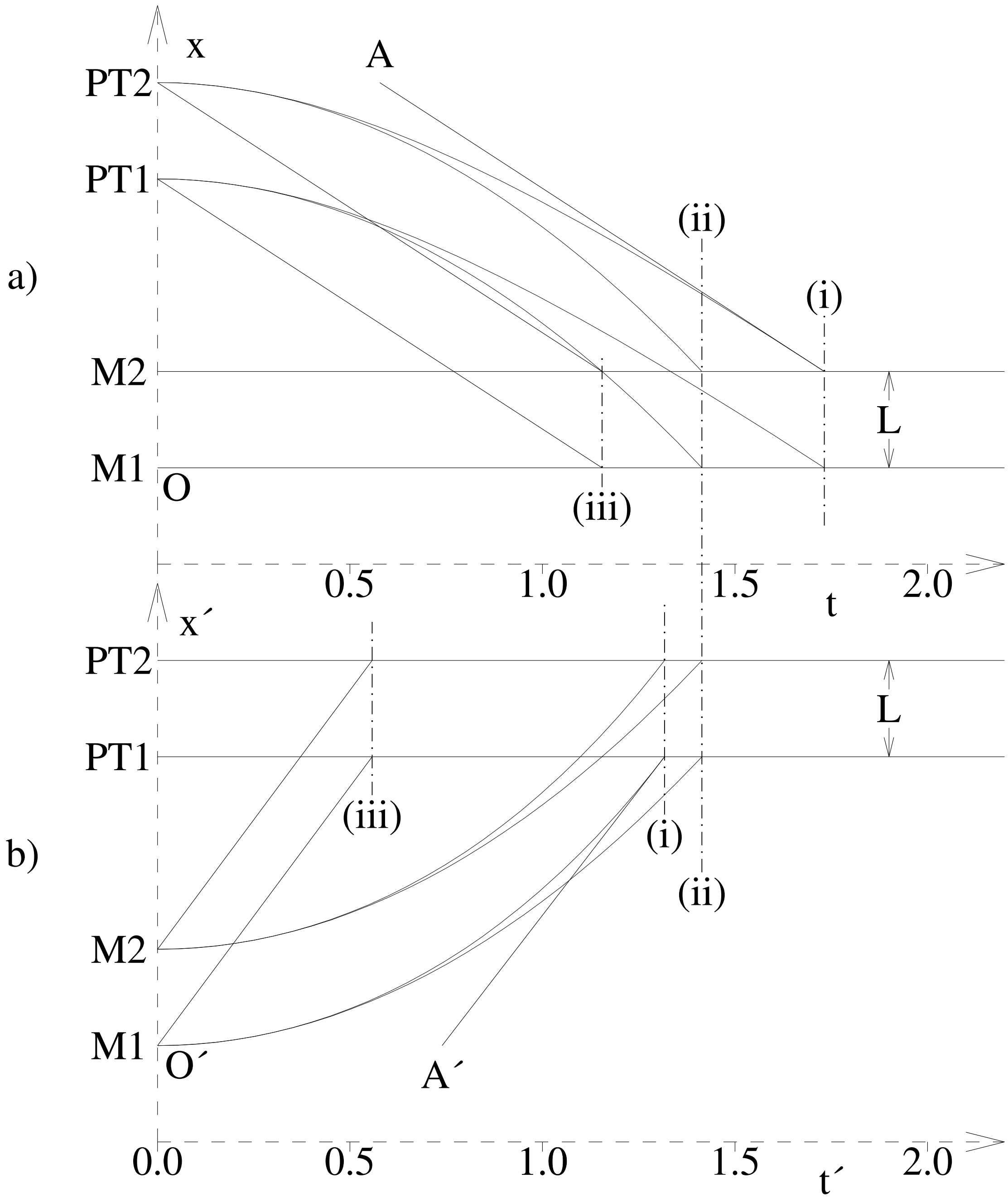

Figure 2: *World lines of PT1, PT2, M1 and M2 in S, a) or S', b) for different acceleration programs and space-time transformation equations.* $c = a_0 = 1$, $L = 1/3$. *Ruler measurements of the spatial separations of PT1 and PT2 or M1 and M2 are indicated by the vertical dot-dashed lines for the three different cases considered. See text for discussion.*

(iii) Uniform motion after impulsive acceleration in special relativity:

$$\begin{aligned} x_{PT1}(t) &= x_{PT2}(t) - L = 3L - vt & (11) \\ x'_{M1}(t) &= x'_{M2}(t) - L = v't' = \gamma vt & (12) \end{aligned}$$

where $\gamma \equiv \sqrt{1-(v/c)^2}$. The constant velocity, v, is chosen according to the equations [10]:

$$\begin{aligned} T &= \frac{c}{a_0}\sqrt{\left(1+\frac{3a_0 L}{c}\right)-1} & (13) \\ v &= -\frac{a_0 T}{\sqrt{1+\left(\frac{a_0 T}{c}\right)^2}} = -\frac{a_0 T}{\gamma(T)} & (14) \\ v' &= c\sinh\frac{a_0 T'}{c} = a_0 T = -\gamma(T)v(T) & (15) \end{aligned}$$

In this way, PT1 and PT2 have the same velocity after impulsive acceleration as they have when arriving at M1 and M2 in case (i) above. It corresponds to the limits $a_0 \to \infty$, $T \to 0$ in Eqs. (14) and (15) for finite values of $a_0 T$, v and v'.

Since the $c \to \infty$ (Galilean) and $t, t' \to 0$ limit of (7) and (8) are the same, for the choice of units and parameters $c = a_0 = 1$, $L = 1/3$ of Fig. 2, the world lines of PT1,PT2 in S (Fig. 2a) and of M1,M2 in S'(Fig. 2b) are indistinguishable, for $t, t' < 0.5$ between cases (i) and (ii). For case (ii) the shapes of the world lines of M1 and M2 in S' are mirror images of those of PT1 or PT2 in S. The different shapes of the world lines of M1 and M2 in S' to those of PT1 or PT2 in S for case (i) are due to the time dilatation effect. In fact, at corresponding values of t and t', the slopes of the world lines of M1 and M2 in S' are γ times the slopes of of PT1 or PT2 in S [10]. The straight world lines of PT1 or PT2 in S and of M1 and M2 in S' have, due to the time dilatation effect, slopes in the ratio $1 : \gamma$, where $\gamma = 2$ corresponding to $v = c\sqrt{3}/2$ or $c = 1$, $a_0 T = \sqrt{3}$ in Eqs. (14) and (15).

The lines A (A'), which are tangents to the world lines of PT2 in Fig. 2a and M1 in Fig. 2b, for case (i), where they intersect those of M2 and PT1 respectively, are parallel to the world lines of PT1 or PT2 (M1 or M2) for case (iii).

The spatial separation of PT1 and PT2 or M1 and M2 is equal to L at all times in both S and S'. Ruler measurements of this separation for the three different cases in both frames are indicated by the labelled vertical dot-dashed lines. It can be seen in Fig. 2 that the constancy and frame invariance of the spatial separations of PT1 and PT2 or M1 and M2 is a necessary geometrical consequence of the identical shapes of their world lines in each frame in all cases. This identity of shape is, in turn, a necessary consequrnce of the identical nature of the acceleration programs to which they are subjected. The invariance of the separation is also independent of the form of the space-time transformation equations by which the shape of the world lines in S' may be derived from those in S.

The initial configuration of the second thought experiment is shown in Fig. 3a. The measuring rod MR, of length L, is used to set the separations of the ruler-mark objects A_1, B_1, A_2, B_2, A_3 and B_3 of the ruler R. The separations of A_1-B_1, B_1-A_2, A_2-B_2 and A_3-B_2 are set to L, and that of B_2-A_3 to $2L$. The front and back ends of MR (as viewed from the left side of the ruler in Fig. 3a are denoted by F and B respectively. As discussed above, if the ends of the moving object are simultaneously aligned with any two of the mark-objects in the proper frame of the ruler, the length of the moving object is defined to be equal to the separation of the mark-objects in the proper frame of the ruler.

The ruler R is subjected to a similar acceleration program to that in case (i) above, specified by the parameter a_0, in such a sense that, in the proper frame, S' of R, MR is observed to move to the right. The equations describing the world lines of the ends of MR as observed in S', are then similar to (8) above:

$$x'(\mathrm{F}) = x'(\mathrm{B}) - L = \frac{c^2}{a_0}\left[\cosh\frac{a_0 t'}{c} - 1\right] \qquad (16)$$

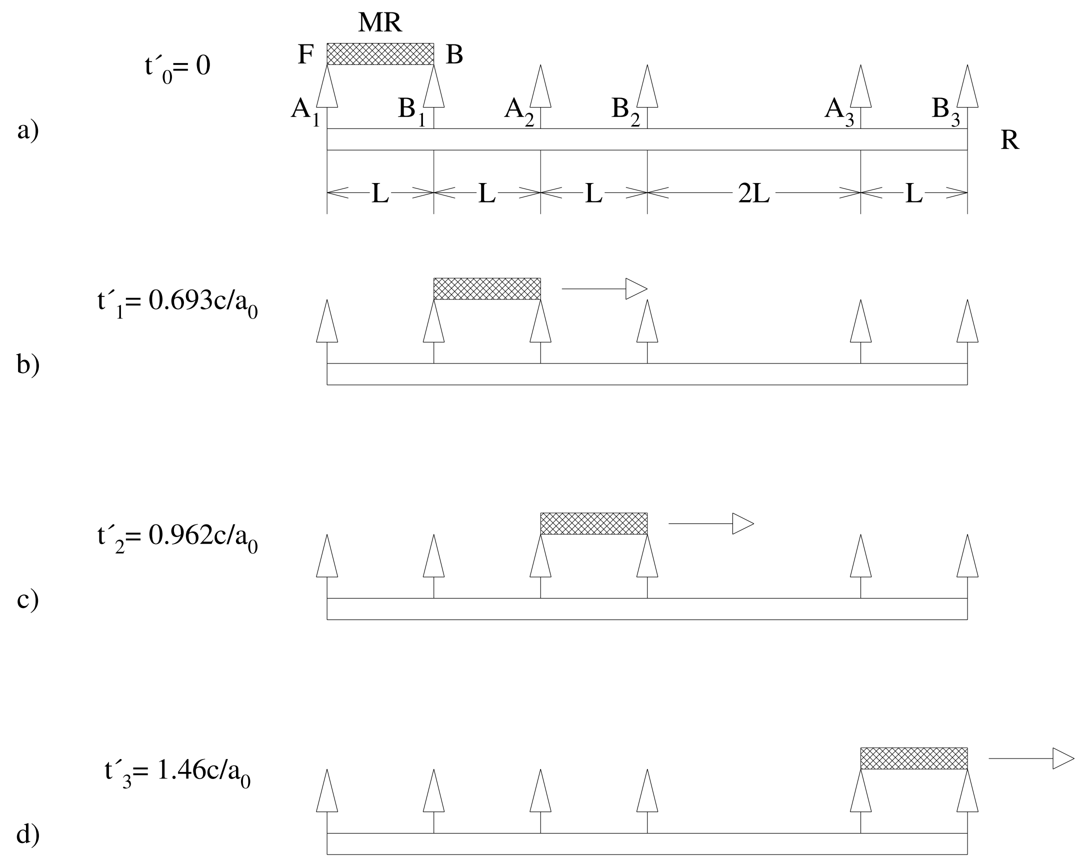

Figure 3: *Four measurements of a stationary measuring rod MR are performed by a moving ruler R. The measurements as observed in the proper frame, S', of R are shown. In a), MR is at rest relative to R. In b) and c) measurements are made while R is accelerating. In d) the ruler moves uniformly relative to MR. In all cases MR is measured to have the same length L —there is no 'length contraction' effect.*

In Fig. 3a, the length of MR at rest, L, is measured by the separation of A_1 and B_1. It follows from (16) that the length of MR is also measured to be L by simultaneous coincidence of F and R with B_1 and A_2 at the epoch $t'_1 = 0.693c/a_0$. In the Figs. 3-5,

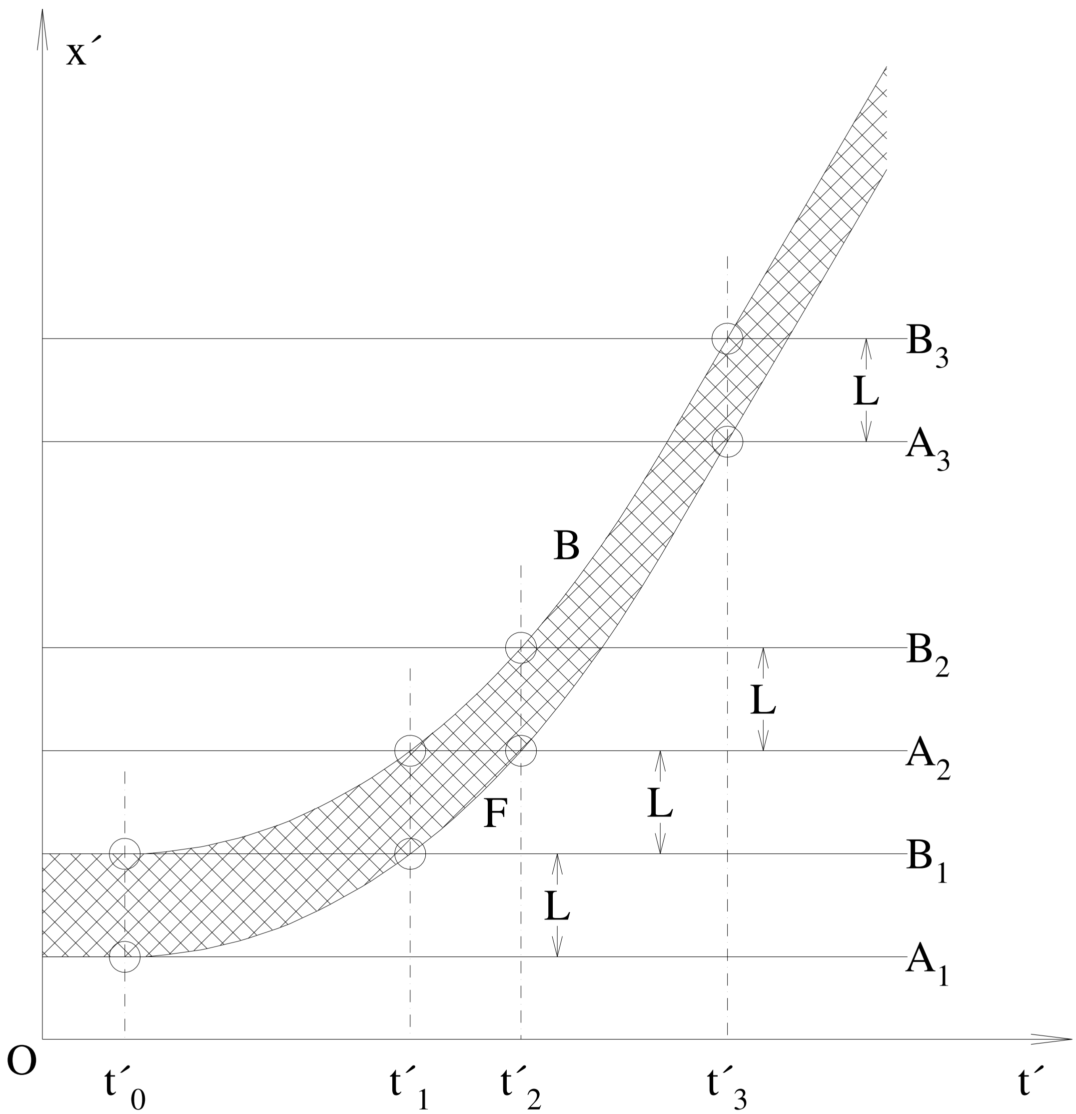

Figure 4: *World lines of the front (F) and back (B) ends of the stationary measuring rod MR, as viewed from the proper frame of the ruler R, showing the epochs* t'_0, t'_1, t'_2 *and* t'_3 *of the four concordant measurements of the length of MR as indicated by the vertical dot-dashed lines.*

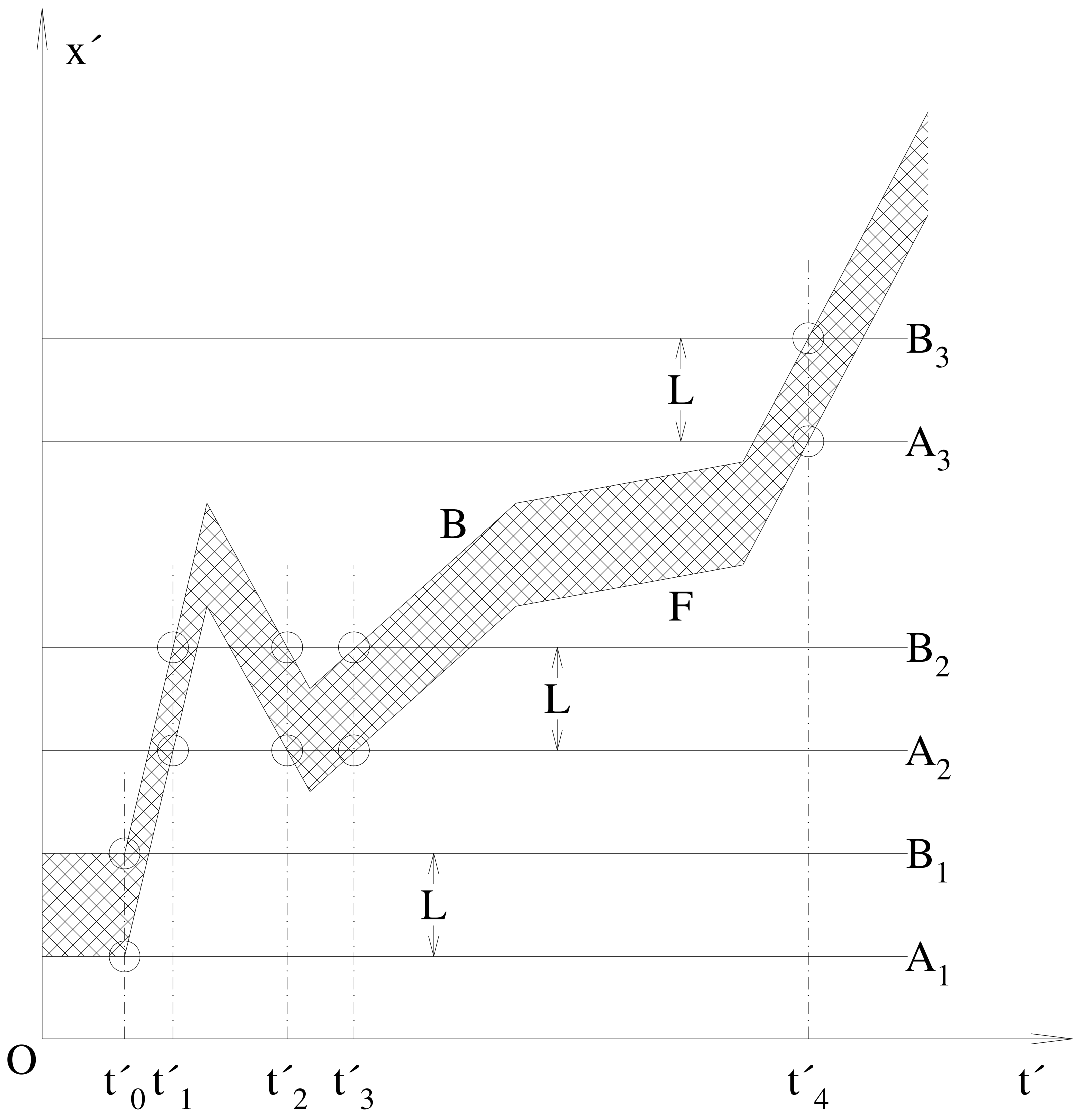

Figure 5: *World lines of the front (F) and back (B) ends of the stationary measuring rod MR, as viewed from the proper frame of the ruler R, when the latter is subjected to a series of impulsive accelerations. Five concordant measurements of the length of MR, indicated by the vertical dot-dashed lines, at epochs* t'_0, t'_1, t'_2, t'_3 *and* t'_4 *are obtained*

units and dimensions are chosen so that $a_0 = c = 4L = 1$. Similar measurements are performed by A_2 and B_2 at $t'_2 = 0.962c/a_0$ and by A_3 and B_3 at $t'_3 = 1.46c/a_0$. As in Ref. [10], it is assumed that the acceleration halts when $x'(\mathrm{F}) = c^2/a_0$ at the epoch in S':

$$t'_{acc} = \frac{c}{a_0}\mathrm{arccosh}(2) = 1.317\frac{c}{a_0} \tag{17}$$

so that for epochs $t' \geq t'_{acc}$, S' is an inertial frame and v' is constant.

The world lines of F and B and the four measurements of the length of MR —one at rest, two during accelerated motion and one during uniform motion— are plotted in Fig. 4. Again, each concordant ruler measurement is indicated by a vertical dot-dashed line. This figure shows clearly that the physical basis for the equality of all the length measurements is that that the first member of (16) may be transposed to give:

$$x'(\mathrm{B}, t') - x'(\mathrm{F}, t') = L \tag{18}$$

where the t' dependence of each term is shown explicitly. Thus the world line of B is derived from that of F by displacing it by the distance L along the positive x'-axis. The operation $x' \to x' + L$ corresponds to moving the origin of coordinates by a distance L. The shape of the world line of an end of MR is invariant under this transformation, and so manifests translational invariance.

For definiteness, the case of 'hyperbolic motion' of the ruler, according to Eqn(16), was considered above, but it is clear that the invariance of length measurements of MR must be independent of the acceleration program of R. In Fig. 5, for example, a series of impulsive accelerations of sign $+, -, +, +$ are applied, yielding equal length measurements at the epochs t'_0, t'_1, t'_2, t'_3 and t'_4 as shown. One measuement corresponds to a simultaneous coincidences of F-A_1 and B-B_1 three to simultaneous coincidences F-A_2 and B-B_2, and one to a simultaneous coincidence of F-A_3 and B-B_3.

Discussion of the reason for the spurious nature of the correlated 'length contraction' and 'relativity of simultaneity' effects of conventional special relativity [11] may be found in Refs. [1, 2, 3, 4, 5].

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