Spatial coordinate systems and relativistic transformation equations

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Abstract

Constants of integration appearing in Lorentz transformation equations relating an event (x, y, z, t) in an inertial frame S to the corresponding event (x', y', z', t') in another inertial frame S' are considered. It is shown that, in the usual application of the Lorentz transformations, different x-coordinate systems are used to specify the positions of two spatially-separated objects. The 'length contraction' and 'relativity of simultaneity' effects of conventional relativity theory are a consequence of this trivial mathematical error

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The history of space-time transformation equations may be said to have begun in 1887 when Voigt [1] noticed that, by modifying the definition of time according to the equation: $t' = t - vx/V^2$, the wave equation: $\partial^2\phi/\partial x^2 = (1/V^2)\partial^2\phi/\partial t^2$ has the same characteristic speed, V, both in an arbitary inertial frame, and in another such frame moving with speed v relative to it. Later Lorentz identified the epoch, t, with that recorded by a clock at rest relative to the aether (the frame in which light is postulated to propagate isotropically with speed c), and t', which he called 'local time', with that recorded by a clock moving with speed v relative to the aether [2]. In the same paper he showed, by applying the transformation to local time, that, to first order in v/c, Maxwell's electromagnetic equations are the same in the aether frame and the frame moving with speed v relative to it.

Later first Larmor [3, 4], then both Lorentz [5] and Poincaré[6], discovered that Maxwell's equations are the same, to all orders in v/c, in the aether, and any other inertial frame, providing that the space and time coordinates appearing in them are transformed according to the 'Lorentz Transformation' (LT) equations:

$$x' = \gamma(x - vt), \quad t' = \gamma(t - vx/c^2), \quad y' = y, \quad z' = z \tag{1}$$

where the space-time coordinates: $(x,\ y,\ z,\ t)$ refer to the aether frame and $(x',\ y',\ z',\ t')$ to a frame moving with speed v relative to the aether and $\gamma \equiv 1/\sqrt{1-(v/c)^2}$. A corollary of this result is that, if light is identified with the electromagnetic waves of Maxwell's theory, as discovered by Hertz [7], the speed of light must be the same in all inertial frames.

The same invariance of Maxwell's equations under the transformations (1) was shown by Einstein in 1905 in the seminal paper on Special Relativity (SpR) [8] where the LT equations were also derived by *postulating* that the speed of light is the same in all inertial frames. In Ref [8] Einstein introduced another interpretation of the LT. For Lorentz and Poincaré, as well as for Larmor, they were a purely mathematical ansatz that had the interesting property of leaving invariant the form of Maxwell's equations. In this calculation no consideration was given to the operational physical meaning of the coordinate symbols in the equations. Einstein assumed that also the LT connected an 'event' $(x,\ y,\ z,\ t)$ observed in one frame to the *same event* $(x',\ y',\ z',\ t')$ as observed in another frame. The very different nature of these two interpretations was pointed out and discussed by Swann [9]. It is the second interpretation, as 'event transformation', that is the subject of the present paper.

The event $(x,\ y,\ z,\ t)$ is observed in an inertial frame S, while the corresponding event $(x',\ y',\ z',\ t')$ is observed in an inertial frame S' moving with speed v along the common x,x' axis of the two frames. It is further assumed that the event lies on the world line of an object (for example a clock) that is at rest in S'. The epoch t' is that recorded by such a clock, while t is recorded by an identical clock at rest at an arbitary position in the frame S. These definitions give immediately strong constraints on both the positions of coordinate origins and the possible values of the spatial coordinates in the LT (1).

Since the object is at rest in S', x' must be independent of time. The right side of the space transformation equation $x' = \gamma(x-vt)$ vanishes, for any value of x, at a certain time $t = x/v$. Since it must also vanish for any other value of t, due to the time-independence of x' on the left side of the equation, the equation *is only valid if both* $x' = 0$ *and* $x = vt$. These equations are simply the (correlated) equations of the worldlines of the object in the frames S' and S respectively. It is then also clear that the physical meaning of the space transformation equation in (1) is the same as that of any equation of the form: $x' = g(x - vt) = 0$ where g is any finite constant or function of v. Setting $g = 1$ gives the 'Galilean' transformation $x' = x - vt = 0$. An alternative way to show that $x' = 0$ in the space LT in (1) is by consideration of the differential form: $dx' = \gamma(dx - vdt) = 0$. The interval dx' vanishes since the object is at rest in S'. Integration of the differential form gives, in general:

$$x'(t) - x'(t_0) = \gamma[x(t) - x(t_0) - v(t - t_0)] = 0. \tag{2}$$

In this expression the x, x' coordinate origins as well as the clock offset, t_0, are arbitary. This reduces to the standard form $x'(t) = \gamma(x(t) - vt)$ if, and only if,

$$x'(t_0) = x(t_0) = 0, \qquad t_0 = 0$$

in which case, as previously shown, $x'(t) = 0$ and $x(t) = vt$. The standard form of the coordinate space LT then corresponds to a particular position, in the frame S', of the object described: at the origin of x' coordinates, and of coordinate system in the frame S: the object is at the x coordinate origin at time $t = 0$.

In the standard text book derivation of 'length contraction' from the LT (originally due to Einstein in Ref. [8]) it is assumed that for some object: $x'(t) \neq 0$, $x'(t_0) = x(t_0) = 0$. i.e that:

$$x'(t) = \gamma[x(t) - vt] \neq 0. \tag{3}$$

These initial conditions already contain a contradiction. Since x' is time-independent it is impossible that (see Eq. (2) above) simultaneously $x'(t_0) = 0$ and $x'(t) \neq 0$ as assumed in (3). It is also clear that (3) is incompatible with the completely general expression (2). Nevertheless, subtracting $x'(0) = x'(t)$ from both sides of (3) gives:

$$x'(t) - x'(0) = \gamma[x(t) - x(0) - vt] = 0 \tag{4}$$

where $x(0) = x'(0)/\gamma$. This means (see Fig. 1a) that in (4), at $t = 0$, the x-coordinate origin is shifted in the positive x-direction by $(\gamma - 1)x'(0)/\gamma$ relative to the x' origin. Notice that, in the above, only coordinate systems, not the spatial separation of discrete objects, have been discussed. However it is essential that in order to measure correctly the spatial separation of two objects *the same coordinate system must be used to specify the position of both objects*. As will now be shown it is the failure to respect this necessary condition that is the origin of the spurious 'length contraction effect' (LCE) of conventional SpR.

As shown in Fig. 1b a second object, labelled A, is placed at $x' = 0$ in the frame S'. The object with $x' \neq 0$ considered above is labelled B. Applying the coordinate space LT to an event on the worldline of A gives the relation:

$$x'(\mathrm{A}) = \gamma[x_{\mathrm{A}}(\mathrm{A}, t) - vt] = 0 \tag{5}$$

so that at $t = 0$, $x_{\mathrm{A}}(\mathrm{A}, 0) = 0$. Thus the x-coordinate x_{A} specifying the position of A has its origin at the position of the object at $t = 0$. In contrast (see Fig. 1a) the x-coordinate $x_{\mathrm{B}}(\mathrm{B}, t)$ specifying the position of the object B has, at $t = 0$, its origin at distance $x'(\mathrm{B})/\gamma$ from B in the negative x-direction. This means that (see Fig. 1b) at $t = 0$, $x_{\mathrm{B}}(\mathrm{B}, 0) = x'(\mathrm{B})/\gamma$. In the calculation of the spatial separation, L, of A and B in the frame S the same coordinate system must be employed for both objects. Choosing x_{A}, setting $t = 0$, and dropping the time label of the coordinates, it is found, as shown in Fig. 1b, that

$$L_{\mathrm{A}} \equiv x_{\mathrm{A}}(\mathrm{B}) - x_{\mathrm{A}}(\mathrm{A}) = x_{\mathrm{A}}(\mathrm{B}) = x'(\mathrm{B}) = x'(\mathrm{B}) - x'(\mathrm{A}) \equiv L' \tag{6}$$

while if the coordinate x_{B} is employed:

$$L_{\mathrm{B}} \equiv x_{\mathrm{B}}(\mathrm{B}) - x_{\mathrm{B}}(\mathrm{A}) = \frac{x'(\mathrm{B})}{\gamma} + \frac{(\gamma - 1)x'(\mathrm{B})}{\gamma} = x'(\mathrm{B}) = x'(\mathrm{B}) - x'(\mathrm{A}) \equiv L'. \tag{7}$$

Thus $L_{\mathrm{A}} = L_{\mathrm{B}} = L'$. The separation of A and B in the frame S is the same as that in the frame S' that is moving with speed v relative to it —there is no LCE. The LCE found in conventional SpR results from the the erroneous assumption that the correct spatial transformation equations for the world lines of A and B are:

$$x'(\mathrm{B}) = \gamma[x(\mathrm{B}) - vt] \neq 0, \tag{8}$$
$$x'(\mathrm{A}) = \gamma[x(\mathrm{A}) - vt] = 0. \tag{9}$$

Subtracting (9) from (8) gives, for any value of t:

$$L' \equiv x'(\mathrm{B}) - x'(\mathrm{A}) = \gamma(x(\mathrm{B}) - x(\mathrm{A})) \equiv \gamma L. \tag{10}$$

This is the putative LCE. The calculation is equivalent to replacing $x_{\mathrm{B}}(\mathrm{A}) = -(\gamma - 1)x'(\mathrm{B})/\gamma$ in Eq. (7) by $x_{\mathrm{B}}(\mathrm{A}) = 0$ to yield the result of Eq. (10). It is then clear that, in Eq. (10) *a different coordinate system in the frame S is being used to* *specify the position of B to that used for A* —a mathematically nonsensical manner to calculate the separation of the objects in this frame.

Integrating the differential form of the time LT: $dt' = \gamma(dt - vdx/c^2)$ gives a formula, analogous to (2) for the space LT:

$$t' - t'_0 = \gamma\left[t - t_0 - \frac{[v(x(t) - x(t_0))]}{c^2}\right]. \tag{11}$$

Employing the coordinate system $x_{\rm A}$ of Fig. 1b and substituting clocks $\rm C_A$, $\rm C_B$ indicating epochs $t'(\rm A)$, $t'(\rm B)$ for the objects A and B, then the particular choice of integration constants: $t'_0(\rm A) = t'_0(\rm B) = t_0 = 0$, $x_{\rm A}({\rm A}, t_0) = 0$ $x_{\rm A}({\rm B}, t_0) = L$ in (11) yields the equations:

$$t'({\rm B}, t({\rm B})) = \gamma\left[t({\rm B}) - v\frac{(x_{\rm A}({\rm B}) - L)}{c^2}\right], \tag{12}$$

$$t'({\rm A}, t({\rm A})) = \gamma\left[t({\rm A}) - v\frac{x_{\rm A}({\rm A})}{c^2}\right]. \tag{13}$$

The same choice of integration constants in the solutions of the differential equation of motion $dx = vdt$ of either clock in the frame S gives:

$$x_{\rm A}({\rm B}) = vt({\rm B}) + L, \tag{14}$$

$$x_{\rm A}({\rm A}) = vt({\rm A}). \tag{15}$$

Combining (12) with (14) and (13) with (15), so as to eliminate the spatial coordinates, yields the time dilation relations:

$$t'({\rm B}, t({\rm B})) = \frac{t({\rm B})}{\gamma}, \tag{16}$$

$$t'({\rm A}, t({\rm A})) = \frac{t({\rm A})}{\gamma}. \tag{17}$$

Since then $t'({\rm B}, t) = t'({\rm A}, t) = t/\gamma$ when $t(A) = t(B) = t$ it is clear that the synchronisation of the clocks $\rm C_A$ and $\rm C_B$ described by (12) and (13) when $t({\rm B}) = t({\rm A}) = 0$ is valid for all later values of t —there is no 'relativity of simultaneity' effect (RSE). Since the synchronistion of $\rm C_A$ and $\rm C_B$ at any particular instant is a purely mechanical or electronic procedure [a] there is no practical difficulty in setting up clock offsets and coordinate systems to obtain an experimental configuration with initial conditions described by Eqs. (12) and (13).

The RSE arises when it is assumed that the 'standard' time LT: $t' = \gamma(t - vx/c^2)$ is equally applicable to both clocks $\rm C_A$ and $\rm C_B$. in this case Eq. (12) above is replaced by:

$$\tilde{t}'({\rm B}, t({\rm B})) = \gamma\left[t({\rm B}) - v\frac{x_{\rm A}({\rm B})}{c^2}\right]. \tag{18}$$

For comparison of this equation with (12) and the general transformation time equation (11), the quantity $\tilde{t}'_0 \equiv -\gamma vL/c^2$ is subtracted form both sides of (18) to give:

$$\tilde{t}'({\rm B}, t({\rm B})) - \tilde{t}'_0 = \gamma\left[t({\rm B}) - v\frac{(x_{\rm A}({\rm B}) - L)}{c^2}\right]. \tag{19}$$

[a] A discussion of different methods to synchronise spatially-separated clocks, without employing light signals, may be found in [10].

When $t(\mathrm{B}) = 0$ then, as shown in Fig. 1b, $x_{\mathrm{A}}(\mathrm{B}) = L$, which implies, according to (19) that $\tilde{t}'(\mathrm{B},0) = \tilde{t}'_0 = -\gamma v L/c^2$. Since, from Eqs. (13) and (15), $t'(\mathrm{A},0) = 0$ is is clear that the clock $\mathrm{C_A}$ as described by (13) and $\mathrm{C_B}$ as described by (18) are *unsynchronised by construction*:

$$\tilde{t}'(\mathrm{B},0) - t'(\mathrm{A},0) = -\frac{\gamma v L}{c^2}. \tag{20}$$

The standard text book derivation of the RSE is obtained by setting $t(\mathrm{A}) = t(\mathrm{B}) = t$ (simultaneous events in the frame S) and subtracting Eq. (13) from Eq. (18) to yield:

$$\tilde{t}'(\mathrm{B},t) - t'(\mathrm{A},t) = -\frac{\gamma v}{c^2}[x_{\mathrm{A}}(\mathrm{B}) - x_{\mathrm{A}}(\mathrm{A})] = -\frac{\gamma v L}{c^2}. \tag{21}$$

The comparision of (20) and (21) shows that the right side of (21), which is conventionally interpreted as a physical RSE effect, is in fact a trivial consequence of the fact that the initial conditions corresponding to Eq. (18) for the clock $\mathrm{C_B}$ necessarily, in conjunction with Eq. (13) for the clock $\mathrm{C_A}$, imply that the two clocks are unsynchronised.

The transformation equations (12)-(15) correctly describing synchronised, spatially separated, clocks are related to the 'standard' LT equations (1) by including additive constants on the right sides of the latter equations:

$$x' = \gamma(x - vt) + X, \tag{22}$$

$$t = \gamma\left(t - \frac{vx}{c^2}\right) + T \tag{23}$$

where with coordinates x', x_{A} as in Fig. 1b:

$$X(\mathrm{A}) = 0, \quad T(\mathrm{A}) = 0, \quad X(\mathrm{B}) = L(1-\gamma), \quad T(\mathrm{B}) = \frac{\gamma v L}{c^2}.$$

As discussed in Ref [11], the necessity to include additive constants such as X and T to correctly describe a synchronised clock, not situated at the origin of coordinates in the frame S', was clearly stated by Einstein in the original SpR paper but this was never done, to the best knowledge of the present writer, before the work presented in Ref. [12].

Arguments similar to those of the present paper were previously given in Ref. [13].

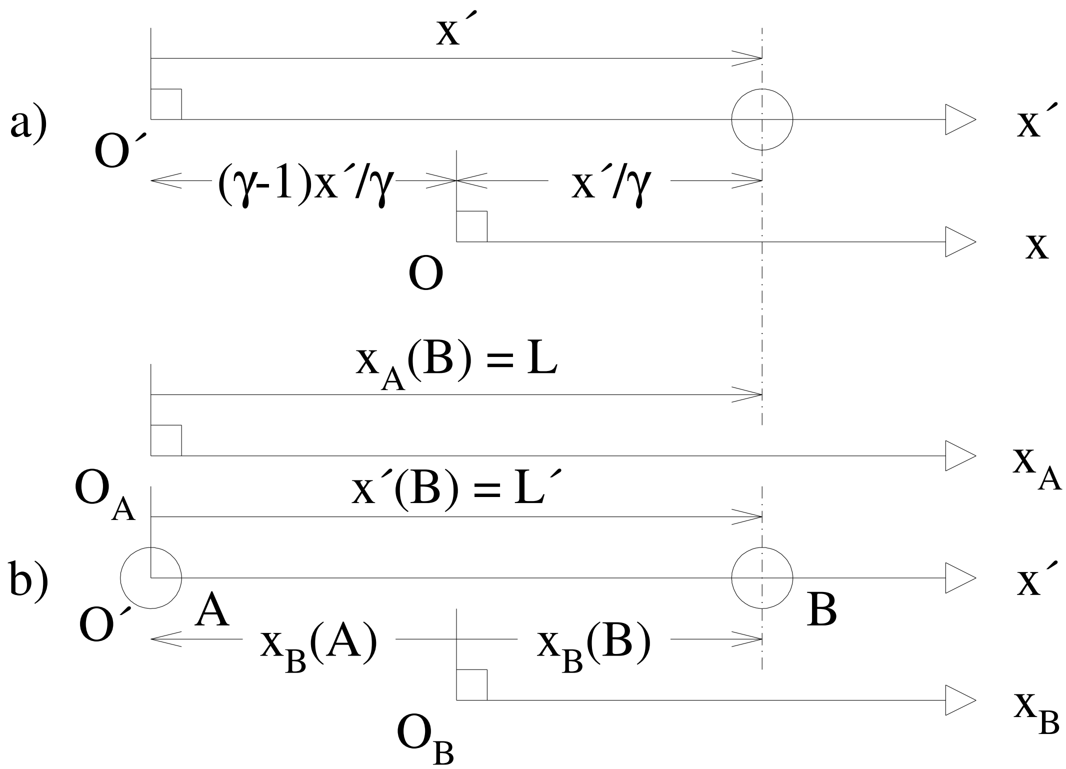

Figure 1: *a) Coordinate systems in the frame S (x) and S' (x') specifying the position , at* $t = 0$*, of an object, with* $x' > 0$*,* $y' = z' = 0$*, at rest in the frame S', according to Eq. (3) or (4). b) Coordinate systems in frame S and S' specifying the position of an object A (at* $x' = y' = z' = 0$*) and B at the position:* $x' = L'$*,* $y' = z' = 0$*, at* $t = 0$*. The system* x_A *has origin coincident with object A, corresponding to the transformation equation (5); the system* x_B *is the same as that shown in a).*

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